ORBITAL MOTION IN MULTIPOLE FIELDS VIA MULTISCALE DECOMPOSITION

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Abstract

We present applications of methods of nonlinear local harmonic analysis in the variational set-up for a description of multiresolution representations in polynomial approximations for nonlinear motions in arbitrary n-pole fields. Our approach is based on the methods allowed to consider the best possible dynamical beam/particle localization in phase space and provided exact multiscale represenstions via nonlinear high-localized eigenmodes for all observables with exact control of contributions to motion from each underlying hidden scale.

INTRODUCTION

In this paper, we consider the applications of a numerical-analytical technique based on the methods of local nonlinear harmonic analysis [1] (in the particular case of underlying affine group a.k.a. wavelet analysis) to the calculations of orbital motion in arbitrary n-pole fields. Our main generic examples here are orbits in transverse plane for a single particle in a circular magnetic lattice in case when we take into account multipolar expansion up to an arbitrary finite number, and particle/beam motion in storage rings [2]. We reduce the complicated initial dynamical problem to a finite number of algebraical problems and represent all dynamical variables via multiscale expansions in the bases of modes maximally localized in the phase space. Our methods here are based on our general universal variational-wavelet approaches considered in papers [3]. Starting in next section from Hamiltonian of orbital motion in magnetic lattice and rational approximation of classical motion in storage rings, in the subsequent part we consider very flexible variational-biorthogonal formulation for a dynamical system with rational nonlinearities and construct the explicit representations for all dynamical variables as expansions in the bases/frames of proper nonlinear high-localized eigenmodes.

MOTION IN MULTIPOLAR FIELDS

The magnetic vector potential of a magnet with $2n$ poles in Cartesian coordinates is

$$A = \sum_n K_n f_n(x, y), \tag{1}$$

where f_n is a homogeneous function of x and y of order n. The cases from $n = 2$ to $n = 5$ correspond to low-order multipoles: quadrupole, sextupole, octupole, decapole. The corresponding Hamiltonian is (ref. [2] for designation):

$$H(x, p_x, y, p_y, s) = \frac{p_x^2 + p_y^2}{2} +$$

$$\left(\frac{1}{\rho^2(s)} - k_1(s)\right)\cdot\frac{x^2}{2} + k_1(s)\frac{y^2}{2} \tag{2}$$

$$-\mathcal{R}e\left[\sum_{n\geq 2}\frac{k_n(s) + i j_n(s)}{(n+1)!}\cdot(x+iy)^{(n+1)}\right].$$

Then we may take into account an arbitrary but finite number of terms in expansion of RHS of Hamiltonian (2) and from our point of view the corresponding Hamiltonian equations of motions are not more than nonlinear ordinary differential equations with polynomial nonlinearities and possibly variable coefficients. As the second generic example, we consider the beam motion in storage rings [2]. Starting from Hamiltonian described classical dynamics in storage rings, and using Serret–Frenet parametrization, we have, after standard manipulations with the truncation of power series expansion of square root, the following approximated (up to octupoles) Hamiltonian for orbital motion in machine coordinates:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1 + f(p_\sigma)]}\\
&+p_\sigma - [1 + K_x\cdot x + K_z\cdot z]\cdot f(p_\sigma)\\
&+\frac{1}{2}\cdot[K_x^2 + g]\cdot x^2 + \frac{1}{2}\cdot[K_z^2 - g]\cdot z^2 - N\cdot xz\\
&+\frac{\lambda}{6}\cdot(x^3 - 3xz^2) + \frac{\mu}{24}\cdot(z^4 - 6x^2z^2 + x^4)\\
&+\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right]
\end{aligned} \tag{3}$$

and the corresponding polynomial series expansion for function $f(p_\sigma)$. We consider here only arbitrary polynomial/rational (in terms of dynamical variables) expressions.

BIORTHOGONAL VARIATIONAL APPROACH VIA LOCALIZED MODES

The first main part of our consideration is some variational approach to these problems, which reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. Multiresolution representation [1] is the second main part of our construction. As a result, the solution is parameterized by solutions of two reduced algebraical problems, one is nonlinear and others are linear problems obtained from proper multiresolution/multiscale constructions. Finally, we obtain the exact (fast convergent numerically) multiscale decomposition via high-localized modes, like compactly supported wavelets or wavelet packets. Because the integrand of our (invariant) variational functional is represented by the bilinear form, it seems more reasonable to consider the constructions which take into account

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all advantages of this structure, so we will use the so-called biorthogonal decompositions [4].

Let us consider the symplectically invariant representation for our initial Hamiltonian problems (2), (3). The key ingredient of such a description is an operator S, symplectic form ω, quasicomplex structure J and proper action of operator S on dynamical variables x:

$$S(\omega(J), H, x, \partial/\partial t, \nabla, t)x = -J\dot{x}(t) - \nabla H(x(t)) \qquad (4)$$

which is polynomial in x and (possibly) has arbitrary dependence on time. It provides us with the following invariant variational formulation:

$$\int_0^T < Sx, y > \mathrm{d}t = 0. \qquad (5)$$

The next is based on the some version of multiresolution machinery. The non-abelian affine group of translations and dilations (it is a simple but generic case) which acts as a hidden symmetry on a proper realization of the underlying finctional space provides the contribution to exact multiscale decomposition from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. We consider a r-regular multiresolution analysis (MRA) of $L^2(\mathbf{R}^n)$ which is a sequence of increasing closed subspaces V_j:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ... \qquad (6)$$

satisfying the following properties:

$$\bigcap_{j\in\mathbf{Z}} V_j = 0, \quad \overline{\bigcup_{j\in\mathbf{Z}}} V_j = L^2(\mathbf{R}^n),$$
$$f(x) \in V_j <=> f(2x) \in V_{j+1},$$
$$f(x) \in V_0 <=> f(x-k) \in V_0, \ \forall k \in \mathbf{Z}^n. \qquad (7)$$

There exists a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x) = \varphi(x - k), \ k \in \mathbf{Z}^n\}$ forms a Riesz basis for V_0. The function φ is regular and localized. Let $\varphi(x)$ be a scaling function, $\psi(x)$ is a wavelet function and $\varphi_i(x) = \varphi(x-i)$. Scaling relations that define φ, ψ are

$$\varphi(x) \ = \ \sum_{k=0}^{N-1} a_k \varphi(2x-k) = \sum_{k=0}^{N-1} a_k \varphi_k(2x), \qquad (8)$$
$$\psi(x) \ = \ \sum_{k=-1}^{N-2} (-1)^k a_{k+1} \varphi(2x+k). \qquad (9)$$

Let indices ℓ, j represent translation and scaling, respectively and

$$\varphi_{jl}(x) = 2^{j/2}\varphi(2^j x - \ell), \qquad (10)$$

then the set $\{\varphi_{j,k}\}, k \in \mathbf{Z}^n$ forms a Riesz basis for V_j. The wavelet function ψ is used to encode the details between two successive levels of approximation. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}:

$$V_{j+1} = V_j \bigoplus W_j. \qquad (11)$$

All V_j are spanned by dilation and translations of the scaling function, while W_j are spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$, where

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k). \qquad (12)$$

All expansions which we used are based on the following properties:

$$\{\psi_{jk}\}, \quad j,k \in \mathbf{Z} \quad \text{is a Hilbertian basis of } L^2(\mathbf{R})$$
$$\{\varphi_{jk}\}_{j\ge 0, k\in\mathbf{Z}} \quad \text{is an orthonormal basis for} L^2(\mathbf{R}),$$
$$L^2(\mathbf{R}) = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j}. \qquad (13)$$

According to machinery [4] we start with two hierarchical sequences of approximations spaces:

$$\begin{array}{c} \ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots, \\ \ldots \widetilde{V}_{-2} \subset \widetilde{V}_{-1} \subset \widetilde{V}_0 \subset \widetilde{V}_1 \subset \widetilde{V}_2 \ldots, \end{array} \qquad (14)$$

and the corresponding biorthogonal expansions:

$$x^N(t) = \sum_{r=1}^N a_r \psi_r(t), \quad y^N(t) = \sum_{k=1}^N b_k \tilde{\psi}_k(t). \qquad (15)$$

Let W_0 be complement to V_0 in V_1, but not necessarily orthogonal complement. Orthogonality conditions have the following form:

$$\widetilde{W}_0 \perp V_0, \quad W_0 \perp \widetilde{V}_0, \quad V_j \perp \widetilde{W}_j, \quad \widetilde{V}_j \perp W_j.$$

The translates of ψ span W_0, the translates of $\tilde{\psi}$ span $\widetilde{W}_0$. Biorthogonality conditions are

$$< \psi_{jk}, \tilde{\psi}_{j'k'} >= \int_{-\infty}^{\infty} \psi_{jk}(x) \tilde{\psi}_{j'k'}(x) \mathrm{d}x = \delta_{kk'}\delta_{jj'}, \quad (16)$$

where $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$. Functions $\varphi(x), \tilde{\varphi}(x-k)$ form dual pair:

$$< \varphi(x-k), \tilde{\varphi}(x-\ell) >= \delta_{kl}, \qquad < \varphi(x-k), \tilde{\psi}(x-\ell) >= 0.$$

Functions $\varphi, \tilde{\varphi}$ generate a (biorthogonal) multiresolution analysis. The translations of bases functions $\varphi(x-k)$, $\psi(x-k)$ are synthesis functions while $\tilde{\varphi}(x-\ell), \tilde{\psi}(x-\ell)$ are analysis functions. Synthesis functions are biorthogonal to analysis functions. Scaling spaces are orthogonal to dual wavelet spaces. Two multiresolutions are intertwining

$$V_j + W_j = V_{j+1}, \quad \widetilde{V}_j + \widetilde{W}_j = \widetilde{V}_{j+1}.$$

These are direct sums but not orthogonal sums. So, our representation (15) for the solution on the level of resolution V_j has now the form

$$x_j(t) = \sum_k \tilde{a}_{jk} \psi_{jk}(t), \qquad (17)$$

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where synthesis wavelets are used to synthesize the decomposition. But $\tilde{a}_{jk}$ come from inner products with analysis wavelets. Biorthogonality yields

$$\tilde{a}_{jm} = \int x^j(t)\,\tilde{\psi}_{jm}(t)\,\mathrm{d}t. \tag{18}$$

So, we may add this powerful construction, accelerating convergence, to our variational approach [3]. We have the modification only on the level of computing coefficients of reduced nonlinear algebraical system of equations (generic generalized dispersion relations). This biorthogonal construction is more flexible and stable under the action of a large class of operators while orthogonal construction (one hidden scale for multiresolution) is fragile. As a result, all computations are much more simpler and we accelerate the rate of convergence. In all types of Hamiltonian calculations based on some bilinear structures such an approach leads to greater success regarding traditional ones. In numerical modelling we considered (periodic) wavelet families/wavelet packets providing the minimum Shannon entropy property and exponential control of convergence in wide classes of underlying functional spaces. We obtain from (5) the following reduced system of algebraical equations (RSAE) on a set of unknown coefficients a_i of expansions (17):

$$L(S_{ij}, a, \alpha_I, \beta_J) = 0 \tag{19}$$

which are the generalized dispersion relations generating spectra for comlex dynamics on a set of coexistent hidden internal subscales. Here the operator L is the algebraization (after the application of variational procedures) of the initial problem (5). $I = (i_1, ..., i_{q+2})$, $J = (j_1, ..., j_{p+1})$ are multiindexes, by which are labelled α_I and β_I, the other coefficients of RSAE (19):

$$\beta_J = \{\beta_{j_1 \dots j_{p+1}}\} = \int \prod_{1 \le j_k \le p+1} \varphi_{j_k}, \tag{20}$$

$$\alpha_I = \{\alpha_{i_1} \dots \alpha_{i_{q+2}}\} = \sum_{i_1, \dots, i_{q+2}} \int \varphi_{i_1} \dots \dot{\varphi_{i_s}} \dots \varphi_{i_{q+2}},$$

where p (q) is the degree of nominator (denominator) part of operator S (4), $i_\ell = (1, ..., q+2)$, $\dot{\varphi_{i_s}} = \mathrm{d}\varphi_{i_s}/\mathrm{d}t$. Now, when we solve RSAE (19) and determine unknown coefficients from formal expansion (17) we therefore obtain the solution of our initial problem. It should be noted that if we consider only truncated expansion with N terms then we have from (19) the system of $N \times n$ (n is the dimension of x (17)) algebraical equations and the degree of this algebraical system coincides with the degree of the initial differential system. The problem of computations of coefficients α_I, β_J (20) of the reduced algebraical system may be explicitly solved inside such a multiresolution approach. The bases functions $\psi_k(t)$ (15) are obtained via multiresolution expansions (14) and represented by compactly supported wavelets. Because affine group of translations and dilations is inside the approach, this method resembles the action of a microscope. So, according to the procedures described above, we compute contributions to the full multiscale decomposition/spectrum from each underlying hidden scale of the full (multi)resolution tower (or more exactly mathematically, filtration) or from the whole infinite scale of subspaces (6) or (14) of the underlying functional space of states. The solution (in case of the simplest non-abelian affine group of hidden symmetry) has the following form

$$x(t) = x_N^{slow}(t) + \sum_{j \ge N} x_j(\omega_j t), \quad \omega_j \sim 2^j, \tag{21}$$

which corresponds to the full multiresolution expansion in all hidden scales.

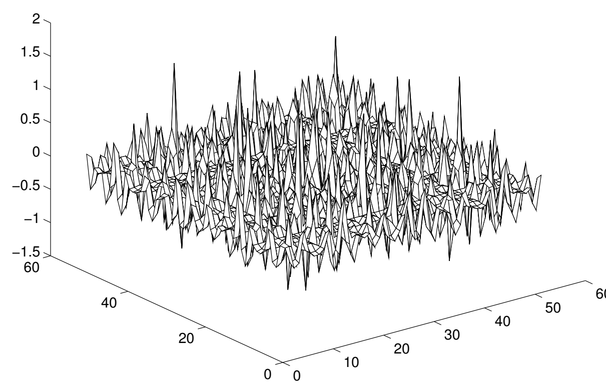

Figure 1: Multiscale representations for $x - p_x$ sections.

Formulas like (21) provide expansion into a slow part x_N^{slow} and fast oscillating parts for arbitrary N. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first term in the RHS of decomposition (21) corresponds on the global level of function space decomposition to the resolution space and the second one to the detail space. In this way we calculate contributions to our final decomposition (21) from each scale of resolution or each time scale. On Fig. 1 we present multiscale representations for the sections $x - p_x$ in phase space corresponding to the models (2), (3), (4) for some (multipole) parameter region. Definitely, such an analysis provides a lot of possibilities to analyze dynamics and to control the quality of particle/beam motion/orbits in accelerators.

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