THE LINEAR PARAMETERS AND THE DECOUPLING MATRIX
FOR LINEARLY COUPLED MOTION IN
6 DIMENSIONAL PHASE SPACE

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Abstract
It will be shown that starting from a coordinate system where the 6 phase space coordinates are linearly coupled, one can go to a new coordinate system, where the motion is uncoupled, by means of a linear transformation. The original coupled coordinates and the new uncoupled coordinates are related by a $6 \times 6$ matrix, $R$. It will be shown that of the 36 elements of the $6 \times 6$ decoupling matrix $R$, only 12 elements are independent. A set of equations is given from which the 12 elements of $R$ can be computed from the one period transfer matrix. This set of equations also allows the linear parameters, the $\beta_i, \alpha_i$, $i=1,3$, for the uncoupled coordinates, to be computed from the one period transfer matrix.

1 THE DECOUPLING MATRIX, $R$

The particle coordinates are assumed to be $x, p_x, y, p_y, z, p_z$. The particle is acted upon by periodic fields that couple the $6$ coordinates. The linearized equations of motion are assumed to be

$$\frac{dx}{ds} = A(s)x$$

$$x = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ z \\ \dot{z} \end{bmatrix}$$

(1-1)

where the $6 \times 6$ matrix $A(s)$ is assumed to be periodic in $s$ with the period $L$. Note that the symbol $x$ is used to indicate both the column vector $x$ and the first element of this column vector. The meaning of $x$ should be clear from the context. The $6 \times 6$ transfer matrix $T(s, s_0)$ obeys

$$x(s) = T(s, s_0)x(s_0)$$

$$\frac{dT}{ds} = A(s)T$$

(1-2)

It is assumed that the motion is symplectic so that

$$TT^T = I, \quad T = STS$$

(1-3)

where $I$ is the $6 \times 6$ identity matrix, $T^T$ is the transpose of $T$ and the $6 \times 6$ matrix $S$ is given by

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

(1-4)

The $6 \times 6$ transfer matrix $T(s, s_0)$ has 36 elements. However, the number of independent elements is smaller because of the symplectic conditions given by Eq. (2-3). There are 15 symplectic conditions or $(k^2 - k)/2$ where $k = 6$. The transfer matrix $T$ then has 21 independent elements.

One can also introduce the one period transfer matrix $\tilde{T}(s)$ defined by

$$\tilde{T}(s) = T(s + L, s)$$

(1-5)

$\tilde{T}(s)$ is also symplectic and has 21 independent elements.

One now goes to a new coordinate system where the particle motion is not coupled. The coordinates in the uncoupled coordinate system will be labeled $u, p_u, v, p_v, w, p_w$. It is assumed that the original coupled coordinate system and the new uncoupled coordinate system are related by a $6 \times 6$ matrix $R(s)$

$$x = Ru$$

$$u = \begin{bmatrix} u \\ p_u \\ v \\ p_v \\ w \\ p_w \end{bmatrix}$$

(1-6)

$R(s)$ will be called the decoupling matrix.

One can introduce a $6 \times 6$ transfer matrix for the uncoupled coordinates called $P(s, s_0)$ such that

$$u(s) = P(s, s_0)u$$

(1-7)

and one finds that

$$P(s, s_0) = R^{-1}(s)T(s, s_0)$$

(1-8)

one can also introduce the one period transfer matrix $\hat{P}(s)$ defined by

$$\hat{P}(s) = P(s + L, s)$$

(1-9)
The decoupling matrix is defined as the $6 \times 6$ matrix that diagonalize $P(s)$, which means here that when the $6 \times 6$ matrix $P$ is written in terms of $2 \times 2$ matrices it has the form
\[
\hat{P} = \begin{bmatrix} \hat{P}_{11} & 0 & 0 \\ 0 & \hat{P}_{22} & 0 \\ 0 & 0 & \hat{P}_{33} \end{bmatrix}
\] (1-10)
where $\hat{P}_{ij}$ are $2 \times 2$ matrices. $\hat{P}$ will be called a diagonal matrix in the sense of Eq. (1-10).

The definition given so far of the decoupling matrix $R$, will be seen to not uniquely define $R$ and one can add the two conditions on $R$ that it is a symplectic matrix and it is a periodic matrix. The justification for the above is given by the solution found for $R(s)$ below.

Because $T(s, s_0)$ and $R(s)$ are symplectic, it follows that $P(s, s_0)$ and $\hat{P}(s)$ are symplectic. Eq. (1-8) can be rewritten as
\[
P(s, s_0) = \mathcal{T}(s) T(s, s_0) R(s_0) \]
\[
\hat{P}(s) = \mathcal{R}(s) \hat{T}(s) R(s) \] (1-11)
It also follows that the $2 \times 2$ matrices has 3 independent elements as $|\hat{P}_{11}| = |\hat{P}_{22}| = |\hat{P}_{33}| = 1$. Eq. (1-12) can be written as
\[
\hat{T}(s) = R(s) \hat{P}(s) \mathcal{R}(s) \] (1-12)
Eq. (1-12) shows that $R$ has 12 independent elements, as $\hat{T}$ has 21 independent elements and $\hat{P}$ has 9 independent elements. As $R$ has only 12 independent elements, one can suggest that $R$ has the form, when written in terms of $2 \times 2$ matrices,
\[
R = \begin{bmatrix} q_1 I & R_{12} & R_{13} \\ R_{21} & q_2 I & R_{23} \\ R_{31} & R_{32} & q_3 I \end{bmatrix}
\] (1-13)
where $q_1, q_2, q_3$ are scalar quantities, the $R_{ij}$ are $2 \times 2$ matrices and $I$ is the $2 \times 2$ identity matrix. The matrix in Eq. (1-13) appears to have 27 independent elements. However, $R$ is symplectic and has to obey the 15 symplectic conditions, and this reduces the number of independent elements to 12. The justification for assuming this form of $R$, given by Eq. (1-13), will be provided below where a solution for $R$ will be found assuming this form for $R$.

Using Eq. (1-13) for $R$ and the symplectic conditions, one can, in principle, solve Eq. (1-12) for $R$ and $\hat{P}$ in terms of the one period matrix $\hat{T}$. This was done by Edwards and Teng[1] for motion in 4-dimensional phase space where $\hat{T}$ has 10 independent elements, $R$ has 4 independent elements and $\hat{P}$ has 6 independent elements. An analytical solution of Eq. (1-12) for the 6-dimensional case was not found. However, a different procedure for finding $\hat{P}$ and $R$ will be given by finding the eigenvectors of $\hat{P}$, using the eigenvectors of the one period matrix, $\hat{T}$.

The $2 \times 2$ matrices $\hat{P}_{11}, \hat{P}_{22}, \hat{P}_{33}$ which make up $\hat{P}$ each have 3 independent elements and can be written in the form
\[
\hat{P}_{11} = \begin{bmatrix} \cos \psi_1 + \alpha_1 \sin \psi_1 & \beta_1 \sin \psi_1 \\ -1/\gamma_1 \gamma \psi_1 & \cos \psi_1 - \alpha_1 \sin \psi_1 \end{bmatrix}
\]
\[
\gamma_1 = (1 + \alpha_1^2)/\beta_1 \] (1-14)
with similar expressions for $\hat{P}_{22}$ and $\hat{P}_{33}$. Eq. (1-14) and the similar expressions for $\hat{P}_{22}, \hat{P}_{33}$ can be used to define the three beta functions $\beta_1, \beta_2$ and $\beta_3$.

## 2 THE LINEAR PARAMETERS $\beta, \alpha, \psi$ AND THE EIGENVECTORS OF THE TRANSFER MATRIX

In this section, the eigenvectors of the one period transfer matrix, $\hat{P}$, will be found and expressed in terms of the linear periodic parameters $\beta, \alpha$ and $\psi$. These will be used below to compute the linear parameters from the one period transfer matrix $\hat{T}$. They will also be used to find the three emittance invariants $\epsilon_1, \epsilon_2$ and $\epsilon_3$ and to express them in terms of the linear parameters $\beta_i, \alpha_i, i = 1, 3$.

The uncoupled transfer matrix obeys
\[
\frac{d}{ds} = P(s, s_0) = B(s) P(s, s_0) \]
\[
B = \mathcal{R} A R + \frac{d\mathcal{R}}{ds} \] (2-1)
This follows from Eq. (1-2) and Eq. (1-11).

One sees from Eq. (2-1) that $B(s)$ is a periodic matrix, $B(s+L) = B(s)$. It can also be shown that $B$ is a periodic, diagonal matrix similar to $\hat{P}$. See [6] for details.

As the $2 \times 2$ matrix $B_{11}$ is periodic, one can show[2] that the eigenvector of the transfer matrix for $\mathcal{T}$ is
\[
\mathcal{v}_1 = \begin{bmatrix} \beta_1^{1/2} \\ \beta_2^{1/2} (\alpha_1 + i) \end{bmatrix} \exp(i\psi_1) \]
\[
\mathcal{v}_1^* S \mathcal{v}_1 = 2i \] (2-2)
with the eigenvalue $\lambda_1 = \exp(i\mu_1)$. $\beta_1(s), \alpha_1(s)$ are periodic functions and the phase function $\psi_1 = \mu_1 s/L + g_1(s)$ where $g_1(s)$ is periodic.

One can now write down the eigenvectors of the $\hat{P}$ matrix using Eq. (2-2). These eigenvectors will be called $u_1, u_2, u_3, u_4, u_5, u_6$, each of which is a $6 \times 1$ column vector with the eigenvalues $\lambda_1 = \exp(i\mu_1), \lambda_3 = \exp(i\mu_2), \lambda_5 = \exp(i\mu_3), \lambda_2 = \lambda_1, \lambda_4 = \lambda_5$ and $\lambda_6 = \lambda_5$.

## 3 COMPUTING THE LINEAR PARAMETERS $\beta, \alpha, \psi$ FROM THE TRANSFER MATRIX

An important problem in tracking studies is how to compute the linear parameters, $\beta, \alpha, \psi$, defined in section 3, from the one period transfer matrix. The one period transfer matrix can be found by multiplying the transfer matrices of each of the elements in a period. A procedure is given below for computing the linear parameters, which also computes the decoupling matrix $R$ from the one period transfer matrix.

The first step in this procedure is to compute the eigenvectors and their corresponding eigenvalues for the one period transfer matrix $\hat{T}$. This can be done using one of the standard routines available for finding the eigenvectors of a real matrix. $\hat{T}$ is assumed to be known. In this case,
there are 6 eigenvectors indicated by the 6 column vectors
\(x_1, x_2, x_3, x_4, x_5\) and \(x_6\). Because \(T^T\) is a real \(6 \times 6\) matrix,
\(x_2 = x_1^*\), \(x_4 = x_3^*\), \(x_6 = x_5^*\). The corresponding eigenvalue for \(x_1\) is \(\lambda_1 = \exp(i\mu_1)\) and the eigenvalue for \(x_2\) is \(\lambda_2^* = \exp(i\mu_1)\). In a similar way, \(\lambda_2, \lambda_3^*\) are the eigenvalues for \(x_3\) and \(x_4\), and \(\lambda_3, \lambda_4^*\) are the eigenvalues for \(x_5\) and \(x_6\). One can show that (see [6] for details).

\[
\begin{align*}
\psi_1 &= ph(x_1) \\
1/\beta_1 &= Im(p_{x1}/x_1) \\
\alpha_1 &= -\beta_1 Re(p_{x1}/x_1)
\end{align*}
\]

where \(Im\) and \(Re\) stand for the imaginary and real part, and \(ph\) indicates the phase.

Using Eq. (3-1), one can find the linear parameters \(\beta_1, \alpha_1\), and \(\psi_1\) from the eigenvector \(x_1\) of \(T^T\). A procedure can be given for computing the entire \(R\) matrix. See [6] for details.

4 THE THREE EMMITANCE INVARIANTS

Three emittance invariants will be found for linear coupled motion in 6-dimensional phase space. Expressions will be found for these invariants in terms of \(\beta_i, \alpha_i\). A simple and direct way to find the emittance invariants is to use the definition of emittance suggested by A. Piwinski[4] for 4-dimensional motion. This is given by

\[
\epsilon_1 = |\tilde{x}_1 S x_1|^2
\]

\(x\) is a \(6 \times 1\) column vector representing the coordinates \(x, p_x, y, p_y, z, p_z\). \(x_1\) is a \(6 \times 1\) column vector which is an eigenvector of the one period transfer matrix \(T\). \(x_1\) is assumed to be normalized so that

\[
\tilde{x}_1 S x_1 = 2i
\]

One first notes that \(\epsilon_1\) given by Eq. (4-1) is an invariant since \(\tilde{x}_1 S x\) is a Lagrange invariant as \(x_1\) and \(x\) are both solutions of the equations of motion. Eq. (3-1) then represents an invariant which is a quadratic form in \(x, p_x, y, p_y, z, p_z\). This result can be expressed in terms of the linear parameters \(\beta_1, \alpha_1\) by evaluating \(\epsilon_1\) in the coordinate system of the uncoupled coordinates. Since the uncoupled coordinates, represented by the column vector \(u\), is related to \(x\) by the symplectic matrix \(R\),

\[
\epsilon_1 = |\tilde{u}_1 S u|^2
\]

\(u_1\) is an eigenvector of the one period matrix \(\hat{P}\), and one sees that because of Eq. (1-11),

\[
x_1 = R u_1
\]

one can now use the result for \(u\), given by Eq. (2-5) and find that

\[
\begin{align*}
\epsilon_1 &= \frac{1}{\beta_1} \left[ (\beta_1 p_u + \alpha_1 u)^2 + u^2 \right] \\
\epsilon_1 &= \gamma_1 u^2 + 2\alpha_1 u p_u + \beta_1 p_u^2 \\
\gamma_1 &= (1 + \alpha_1)^2/\beta_1
\end{align*}
\]

5 REFERENCES