After separating the Vlasov equation, we find the radial functions and calculate (with no approximations), the stabil-
ity integral, and evaluate the complex perturbation Laplace frequency $s$. Hence

$$
\psi_{+m}(J) = \frac{\omega_s^2 \xi n}{s + i n \omega_s} \Psi_0 \sum_p Z_p(\omega, \sigma) \frac{1}{p} \lambda_p J_{+n}(+p\pi k) .
$$

where $k^2 = J/J_0$ and $J_0 = \omega_s \pi^2 /2$ is an action value, but the $J_n(\ldots)$ with an argument are Bessel functions. The prime notation indicates a derivative with respect to action $J$. The Fourier harmonics are:

$$
2\pi \lambda_q(n) = \int \psi_n(J)e^{-i\theta} e^{i+n\theta} d\theta dJ ,
$$

and so the eigenvalue problem is

$$
\lambda_q(n) = \frac{\omega_s \xi n}{s + i n \omega_s} \sum_p Z_p(\omega, \sigma) \frac{1}{p} \lambda_p I_n(q, p) .
$$

Note, in the above equation the $\lambda_q$ without arguments is the sum over the $\lambda_q(n)$ with arguments.

A. Single azimuthal mode and narrowband impedance

Consider the case of a solitary azimuthal $\psi(J, \theta) = \psi_m(J)e^{im\theta}$. Consider the case of a nar-
rowband impedance such that $Z_q$ is only significant in the vicinity of $p = q > 0$. This results in an eigenfrequency equation:

$$
(s + \omega m \lambda) = m\omega_s I_m(q, q) \xi [Z_{+q} - Z_{-q}] / q .
$$

where $I_m(q, q)$ is an action value, but the

$$
\omega/(m\omega_s) = I_m \xi[X(q\omega_{rf} + \omega) + X(q\omega_{rf} - \omega)] / q - 1
$$

$$
\sigma/(m\omega_s) = I_m \xi[R(q\omega_{rf} + \omega) - R(q\omega_{rf} - \omega)] / q .
$$

These equations have to be solved recursively for $s$. At high enough current, there is a solution with mode frequency $s = 0$, which satisfies the condition:

$$
q = 2\xi X(q\omega_{rf}) I_m(q, q) .
$$

B. $\pm m$ mode-coupling and narrowband impedance

Consider the case of two azimuthal modes,

$$
\psi(J, \theta) = [\psi_{+m} e^{+im\theta} + \psi_{-m} e^{-im\theta}] .
$$

REFINEMENTS TO LONGITUDINAL, SINGLE BUNCH, COHERENT INSTABILITY THEORY

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Abstract

For the case of a bunched beam con®ned to a quadratic po-
tential well, we demonstrate the necessity for considering mode-
coupling to correctly obtain the threshold current for the d.c. in-
stability. Further we ®nd the effect upon growth rate and co-
herent tune shift of evaluating the impedance at a complex fre-
quency. For the case of a bunched beam con®ned to a cosine po-
tential well, we give an exact analytic expression for the disper-
sion integral, and calculate (with no approximations), the stabil-
ity diagram for the Robinson instability taking into account Landau damping. This paper comprises extracts from a lengthy inter-
stantial report[1].

I. SIMPLE HARMONIC OSCILLATOR CASE

We consider the stability of a single bunch con®ned in a
quadratic potential well that is truncated at rf-phase $x = \pm \pi$.
Let $\omega_s$ be the synchronous frequency. We shall investigate the
stability of the system through use of the linearized Vlasov equa-
tion in which products of two perturbation terms will be ignored.
Let the phase-space steady-state and perturbation distribution
functions be $\Psi_0(J)$ and be $\Psi_1$, respectively. In action-angle co-
ordinates $(J, \theta)$, the Vlasov equation becomes:

$$
[\partial/\partial t + (d\theta/\partial t) \partial/\partial \theta] \Psi_1 = (\partial \Psi_0/\partial J)(\partial H/\partial \theta) .
$$

(1)

We shall assume $\Psi_1$ to have time dependence $e^{st}$ with the
complex perturbation Laplace frequency $s = \sigma + i \omega$. Let
$i = \sqrt{-1}$ and take $R[[\ldots]]$ to mean "form the real part".

Henceforward, we shall employ the symbols $q$ and $p$ as integer
indices for Fourier harmonics.

Let $\xi = 2\pi I_{d.c.} / V_{hf}$.

The beam current perturbation signal is $\lambda(x, t) =
R[e^{st} \sum q \lambda_q e^{iq\theta}]$ and leads to perturbing forces $\partial H/\partial \theta =
\omega_s^2 w(x, t)$ where the wake†elds are:

$$
w(x, t) = \xi R[e^{st} \sum q u_q] \quad \text{with} \quad u_q = Z_q(\omega, \sigma) \lambda_q e^{iq\theta} .
$$

(2)

The arguments of the complex impedance $Z_q$ are used to indicate
the modulation sideband frequency. Hence $Z_{\pm q}(\pm \omega, \sigma) =
Z(\pm q\omega_{rf} + \omega, \sigma)$ and $Z_{\pm q}(\pm q\omega_{rf} + i \sigma) \pm i \sigma)$ is the complex impedance evaluated at the $\omega - i \sigma$ sideband of the $q$th harmonic of the radio-

frequency $\omega_{rf}$.

As a trial solution of the Vlasov equation we take

$$
\Psi_1 = R[\psi e^{st}] \quad \text{with} \quad \psi(J, \theta) = \sum_m \psi_m(J) e^{im\theta} ;
$$

(3)

where $m$ is the azimuthal mode index, and $m = 0$ is excluded. After separating the Vlasov equation, we ®nd the radial functions

$$
\psi_{+m}(J) = \frac{\omega_s^2 \xi n}{s + i n \omega_s} \Psi_0 \sum_p Z_p(\omega, \sigma) \frac{1}{p} \lambda_p J_{+n}(+p\pi k) .
$$

(4)

where $k^2 = J/J_0$ and $J_0 = \omega_s \pi^2 /2$ is an action value, but the $J_n(\ldots)$ with an argument are Bessel functions. The prime notation indicates a derivative with respect to action $J$. The Fourier

$$
2\pi \lambda_q(n) = \int \psi_n(J)e^{-i\theta} e^{i+n\theta} d\theta dJ ,
$$

and so the eigenvalue problem is

$$
\lambda_q(n) = \frac{\omega_s \xi n}{s + i n \omega_s} \sum_p Z_p(\omega, \sigma) \frac{1}{p} \lambda_p I_n(q, p) .
$$

(6)

Note, in the above equation the $\lambda_q$ without arguments is the sum over the $\lambda_q(n)$ with arguments.

A. Single azimuthal mode and narrowband impedance

Consider the case of a solitary azimuthal $\psi(J, \theta) = \psi_m(J)e^{im\theta}$. Consider the case of a nar-
rowband impedance such that $Z_q$ is only significant in the vicinity of $p = q > 0$. This results in an eigenfrequency equation:

$$
(s + \omega m \lambda) = m\omega_s I_m(q, q) \xi [Z_{+q} - Z_{-q}] / q .
$$

(7)

where $I_m(q, q)$ is an action value, but the

$$
\omega/(m\omega_s) = I_m \xi[X(q\omega_{rf} + \omega) + X(q\omega_{rf} - \omega)] / q - 1
$$

$$
\sigma/(m\omega_s) = I_m \xi[R(q\omega_{rf} + \omega) - R(q\omega_{rf} - \omega)] / q .
$$

(10)

These equations have to be solved recursively for $s$. At high enough current, there is a solution with mode frequency $s = 0$, which satisfies the condition:

$$
q = 2\xi X(q\omega_{rf}) I_m(q, q) .
$$

(11)

B. $\pm m$ mode-coupling and narrowband impedance

Consider the case of two azimuthal modes,

$$
\psi(J, \theta) = [\psi_{+m} e^{+im\theta} + \psi_{-m} e^{-im\theta}] .
$$

(12)
Consider the case that impedance $Z_p$ is only sign@cant in the vicinity of $p = q > 0$. This results in an eigenfrequency equation.

$$\frac{i(s^2 + m^2 q^2)}{(m^2 \omega_s)^2} = 2\xi[Z(+q \omega t + \omega, \sigma) - Z(-q \omega t + \omega, \sigma)] \frac{q}{1} I_m(q, q).$$

(13)

The equation separates into imaginary and real parts as:

$$\frac{\omega^2 - \sigma^2}{(m^2 \omega_s)^2} = 1^2 - 2I_m \xi[X(q \omega t + \omega) + X(q \omega t - \omega)]/\sigma$$

$$\omega \sigma/(m^2 \omega_s) = +I_m \xi[R(q \omega t - \omega) - R(q \omega t + \omega)]/\sigma.$$  

(14)

At high enough current, there is a solution with mode frequency $s^2 \equiv 0$.

$$q = 4\xi X(q \omega t) I_m(q, q).$$

(15)

The value of the threshold differs by a factor 2 from the case of no mode coupling, expression (11).

II. IMPEDANCE AT COMPLEX FREQUENCY

If we continue $Z$ into the complex plane, given the functional form $Z(\omega, 0)$, then the response to $\exp(\sigma + i \omega t)$ is $Z(\omega, \sigma) = Z(\omega - i \sigma)$. Actually, one does not need to know the form, but only the derivatives of resistance $R$ and reactance $X$ with respect to frequency $\omega$. We denote derivatives with respect to real angular frequency $\omega$ by $\partial_{\omega}$. Let $Z(\omega, \sigma) = R + iX$. We may then employ the Cauchy-Riemann conditions for analytic complex functions:

$$\partial R/\partial \omega = -\partial X/\partial \omega \quad \text{and} \quad \partial X/\partial \omega = +\partial R/\partial \omega,$$

to nd the @rst order Taylor expansion

$$Z(\omega', \sigma') \approx Z(\omega', 0) + (\omega' i \sigma') \times (R, X)\bigg|_{\omega = \omega'}.$$  

(16)

A. Eigenvalues with narrowband impedance

Consider a narrowband impedance that is still suf®ciently broad to include both the upper and lower sideband. An approxima- tion of $|Z_{-q} - Z_{+q}|$ is

$$Z_{-q}(\omega, \sigma) - Z_{+q}(\omega, \sigma) \approx -2iX(q \omega t) + (\omega - i \sigma) \partial_{\omega} R(q \omega t).$$

(17)

Substitution of (17) into (14) leads to the eigenvalue

$$[\omega^2 + \sigma^2]/(m^2 \omega_s)^2 = 1^2 - 4I_m \xi X(q \omega t)/\sigma$$

$$\sigma/(m^2 \omega_s) = -2I_m \xi m s \partial_{\omega} R(q \omega t)/\sigma.$$  

(18)

These forms show that, to @rst order, and for single bunch instability, evaluation of the impedance at a complex frequency alters the coherent tune, but does not change the growth rate.

III. SIMPLE PENDULUM OSCILLATOR

Consider the stability of a single bunch con@ned in a sinusoidal potential well. The unperturbed Hamiltonian is:

$$H(x, y) = y^2/2 + \omega_o^2[1 - \cos x] = y^2/2 + 2\omega_o^2 \sin^2(x/2).$$

(20)

We shall investigate the stability of a multi-particle system of oscillators through use of the Vlasov equation; the equation is simpli®ed if we employ action-angle coordinates.

$$\sin(x/2) = k \sin \theta = \sqrt{J/J_0} \sin \theta$$

$$y = 2\omega_o k \cos \theta = 2\omega_o \sqrt{J/J_0} \cos \theta.$$  

(21)

$J_0 = 2\omega_o$ and $\sin, \cos \theta$ are Jacobean elliptic functions.

The time variation of $\theta$ is $\theta = \omega_o(t - t_0)$ where $t_0$ is a constant of integration.

The trial solution $\psi$ must be separable after integrating $\theta$ over the interval $[-2\kappa, +2\kappa]$. Hence, we take:

$$\Psi_1 = R[e^{it} \sum_{n=-\infty}^{\infty} \psi_m(J) e^{in \pi \theta/2\kappa}]$$

(23)

After separation, we @nd the radial functions $\psi_m$:

$$\psi_m(J) = \frac{\omega_o^2 \xi \sum_{n=0}^{\infty} Z_p(\omega) \lambda_q}{[s + i n \omega_s(J)]} \frac{\pi}{2\kappa} \sum_{p} Z_p(\omega) \lambda_p \frac{\pi}{2} \int_{\kappa}^{\kappa} \frac{\psi_p(J) \psi_q(J)}{s + i n \omega_s(J)} dJ.$$

(24)

Using the Jacobean elliptic analogue of the Hankel transform we @nd a particular case of Lebedev's[2] expression:

$$\lambda_q(n) = \omega_o^2 \xi \sum_{p=0}^{\infty} Z_p(\omega) \lambda_p \frac{\pi}{2} \int_{\kappa}^{\kappa} \frac{\psi_p(J) \psi_q(J)}{s + i n \omega_s(J)} dJ.$$  

(25)

If we sum this equation over mode number $n$, we obtain an eigenvalue problem for the harmonics $\lambda_q$. The form factors are

$$\lambda_q(n) \times \lambda_q(0) = \pi \sqrt{2\kappa} e^{-i\pi \theta/2\kappa} d\theta$$

and have the properties: $\lambda_q(0, 0) = 0$ and $\lambda_q(0, 1) = 0$.

A. Narrowband impedance at cavity radio-frequency

In general, the integrals $\psi_q(J, k)$ are awkward to evaluate analytically. To simplify, we shall consider an impedance that is sign@cant only at the $p = \pm 1$ harmonics of the cavity radio-frequency. For odd-$n$ we @nd:

$$\lambda_q(1, k) = n \frac{\pi \sqrt{2\kappa} q^{n/2}}{1 + q^n} \quad \text{with} \quad q = \exp \left[-\pi \kappa(k')\right].$$

(27)

Here $q$ is the 'nome' and $(k')^2 = 1 - k^2$. Expressions for even-$n$ are rather complicated, but

$$\lambda_q(2, k) \approx \frac{2\pi \sqrt{2\kappa} q^{n/2}}{2(1 + q^n)}.$$  

(28)
B. \( \pm m \) mode coupling and narrowband impedance

Previously, we saw that a mode-coupling theory is essential when the tune shifts and growth rates are comparable with the unperturbed synchrotron frequency. Consequently, we shall not bother to consider the cases of the \(-m\) and the \(+m\) modes in isolation. Let the mode index \( m \) be single sided and valued and take the trial distribution function (12). For the narrowband impedance we obtain the eigenfrequency equation:

\[
i = \xi [Z_{+1} - Z_{-1}] 2m^2 \omega_0^2 \int_0^{J_0} \frac{\omega_s(J) \Psi_0 J_0^m(1, k)}{\omega^2 + m^2 \omega_0^2(J)} dJ .
\]

(29)

Let us search for a threshold and take \( s = i\omega \) pure imaginary. Let the value of action at which the integral is singular be \( J(\omega) \) and definite \( \tilde{k} = \sqrt{J/J_0} \). Then we have the eigenequation:

\[
i = \xi [Z(-\omega_{rf} + \omega) - Z(\omega_{rf} + \omega)] \times [f(\omega) + ig(\omega)]
\]

(30)

where the quantities \( f \) and \( g \) are:

\[
f(\omega) = 2m^2 \omega_0^2 \mathcal{P} \int_0^{J_0} \frac{\omega_s(J) \Psi_0 J_0^m(1, k)}{\omega^2 - m^2 \omega_0^2(J)} dJ
\]

(31)

\[
g(\omega) = \omega_0^2 \pi \Psi_0^m(\tilde{k}) \frac{J_0^m(1, k)}{[\partial \omega_s/\partial J]} f .
\]

(32)

Here \( \mathcal{P} \) indicates the principal value, and \( g \) is the residue.

C. Power limited instability

For the power limited instability, the eigenfrequencies are \( \omega^2 = 0 \). Now zero frequency is either inside the spread of incoherent frequencies, or (for a full bucket) at the very edge of the bunch where there are no particles. Consequently, this particular instability is not Landau damped. We substitute \( \omega^2 = 0 \) and \( \hbar \) and the Fourier components are equal \( \lambda_{-1} (\pm m) = \lambda_{-1} (\pm m) \), and that the threshold current is given by

\[
1 = 4 \xi X(\omega_{rf}) \omega_0^2 \int_0^{J_0} \frac{\Psi_0 J_0^m(1, k)}{\omega_0^2(J)} dJ .
\]

(33)

This only differs from the linear oscillator case by virtue of the exact value of the integral.

D. Stability Diagram for \( m = \pm 1 \) mode coupling

We can generate constraints on the allowable impedance by considering

\[
U(\omega) + iV(\omega) = \frac{i}{[f + ig]} \frac{I_{b.c.}}{I_{b.c.}} \frac{2\pi I_{b.c.}}{V_{rf}} [Z_{-1} - Z_{+1}] .
\]

(34)

Here \( V_{rf} \) is the cavity voltage summed about the ring, and \( U, V \) have been normalized so that the excitation current is independent of bunch length. The method is to plot contours of constant growth rate in the \( U, V \)-plane by scanning \( \omega \). The instability threshold is given by the curve of zero growth rate and is a function of \( J \). If on the same plot, the curve \( Z(\omega) \) lies wholly inside the threshold curve, then that mode is stable. As shown in
Figure 1. Stability diagram for $\alpha = 1/2$ as function of $\dot{J}$.

Figure 2. Stability diagram for $\alpha = 1$ as function of $\dot{J}$.

Figure 3. Stability diagram for $\alpha = 2$ as function of $\dot{J}$.

Figure 4. Stability diagram for $\alpha = 10$ as function of $\dot{J}$.