Optimization of the End Winding Geometry of Dipole Magnets

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Abstract

A simple, almost entirely analytic, method for the optimization of stress-reduced magnet-end winding paths for ribbon-like superconducting cable is presented. This technique is based on characterization of these paths as developable surfaces, i.e. surfaces whose intrinsic geometry is flat. The method is applicable to winding mandrels of arbitrary geometry. Computational searches for optimal winding paths are easily implemented via the technique. Its application to the end configuration of the cylindrical SSC magnet is discussed. The method may be useful for other engineering problems involving the placement of thin sheets of material.

I. INTRODUCTION

A number of superconducting dipole magnet designs and, in particular, the Superconducting Super-Collider (SSC) dipole employ a conductor which has the form of a flat ribbon, which is wound on a cylindrical mandrel. In the body of the magnet the cables run parallel to the axis of the cylinder, thin edge against the cylinder and, usually, in distinct constant theta planes. At the end of the magnet, the cables must loop around over the top of the beam pipe and go into the reflection symmetric location on the opposite side of the pipe for the return journey down the magnet.

Minimization of the stresses caused by this end winding requires that one avoid bending or twisting of the cable in the unfavorable, high moment of inertia, or hard direction, which leads to the concept of a constant periphery design [1] for the end winding.

In the mathematically rigorous sense, this requires that the surface of the ribbon lie in a flat-space with zero Gaussian curvature and mean curvature always normal to the ribbon and in the thin direction. Such surfaces may be visualized with a ribbon of paper which, when smoothly bent without any creasing or tearing, adopts a path with almost no curvature in the high moment of inertia direction. This flat-space model implies a constant periphery and has the property that the high moment of inertia plane is not sheared or twisted.

The intrinsic geometry of the flat-space, developable surface model is always cartesian. The wound surface can always be unwound, and without stretching layed flat on a plane. Such a surface is called "developable". Developable surface models have previously been considered in the SSC magnet end design [2].

Here, we develop this flat-space model analytically and describe the application of the results to path optimization of dipole end windings. In large part, the mathematics involved in our treatment is well known, has a history of over one century, and may be found in texts on differential geometry [3].

II. INTRINSIC GEOMETRY AND COORDINATES

Let $\mathbf{r}(t, h)$ be the three-dimensional position vector to a point on the ribbon surface as a function of two orthogonal arc lengths, $t$ and $h$. $t$ increases in the long direction of the ribbon while $h$ increases in the short direction. As the surface is intrinsically flat, $t$ and $h$ are geodesic coordinates. Let $\mathbf{r}_0(t) = \mathbf{r}(t, h = 0)$ be the vector describing the "base-curve", the intersection of the ribbon surface with the winding mandrel. $\mathbf{r}_0(t)$ is constrained to lie in the surface of the mandrel, for our purposes, the surface of a cylinder. The coordinate system shown in Fig. 1 is defined in terms of the intrinsic Frenet parameters of this base-curve inscribed on the winding mandrel. Define a right-handed triad of unit vectors, the unit tangent,

$$\mathbf{t}(t) = \frac{d\mathbf{r}_0}{dt}(t),$$  \hspace{1cm} (1)

the unit normal $\mathbf{n}(t)$, and the unit binormal $\mathbf{b}(t)$, such that $\mathbf{t} \times \mathbf{n} = \mathbf{b}$. 

Figure 1: Intrinsic coordinate system.
These unit vectors obey the Frenet-Serrat relations,

\[
\begin{align*}
\frac{d\tau}{dt}(t) &= -\kappa(t) \hat{n}(t), \\
\frac{d\hat{n}}{dt}(t) &= \kappa(t) \hat{\tau}(t) + \gamma(t) \hat{b}(t), \\
\frac{d\hat{b}}{dt}(t) &= -\gamma(t) \hat{n}(t).
\end{align*}
\]

(2)

Here, \(\kappa(t) = 1/r(t)\) is the local curvature of the base-curve and \(\gamma(t)\) is the torsion. \(r(t)\) is the local radius of curvature. Also define an important dimensionless quantity \(\rho\), the ratio of the torsion to the curvature,

\[\rho = \frac{\gamma}{\kappa}.\]

(3)

This parameter plays a central role in the exact theory.

At every point of a developable strip at least one of the principal curvatures is zero and there exists a unit vector \(\hat{d}\) which points in the zero curvature direction, that is, along the floor of a valley. In particular, this vector \(\hat{d}(t)\) exists at all points on the base-curve. We assume that the base-curve is such that \(\hat{d}\) is unique and never parallel to \(\hat{n}\). Then one may write the position vector to any point on the strip as

\[
\vec{r}(t, d) = \vec{r}(0) + d \hat{d}(t).
\]

(4)

In contrast to the previous parameterization of the surface, \(t\) and \(d\) are not orthogonal arc lengths but both are still geodesics.

The fact that one of the principle curvatures vanishes allows us to express \(\hat{d}(t)\) in terms of the Frenet-Serrat parameters

\[
\hat{d}(t) = \frac{\hat{n}(t)}{\sqrt{1 + \rho^2}}.
\]

(5)

The flat-space extension of \(\vec{r}(0)\) is completely defined in terms of the intrinsic Frenet parameters of the base-curve via equation 4. This surface is built out of the rectifying planes of the base-curve. These are the planes locally defined by the tangent and binormal vectors. The surface generated in this manner is called the rectifying developable of the base-curve [3].

A straightforward calculation yields the exact principal curvature at any point \((t, d)\) on the ribbon.

\[
\kappa_p = \kappa \left[ \frac{1 + \rho^2}{1 - \hat{\rho} \sqrt{1 + \rho^2}} \right],
\]

(6)

where \(\hat{\rho}\) is the derivative of \(\rho\) with respect to \(t\).

As it stands, equation 4 for the vector position vector of points on the strip is not immediately useful. What we actually desire is the position of the opposite edge of the strip given a position on the base curve \(\vec{r}(0)\). Fortunately, we are dealing with an intrinsically flat space and plane geometry can be used.

Given a point \(t\) on the base curve we seek the position of the opposite edge of the strip, a distance \(h = H\) away in the direction orthogonal to \(t\). Hence, we need to find the point \(t'\) whose null curvature or \(\hat{d}(t')\) vector intercepts the other edge at \((t, H)\) in orthogonal coordinates. The three points, \((t, 0), (t', 0),\) and \((t, H)\) form a right triangle in the plane geometry of the strip and we therefore have that

\[t = t' - H \rho(t').\]

(7)

Equation 7 defines \(t'\) implicitly given \(t\), however since it is nonlinear, it is possible that multiple solutions exist, or that no solutions exist. As written, it can be regarded as an equation for \(t\) given \(t'\). One condition for an unique solution is that the curve be monotonically increasing, i.e.

\[\frac{dt}{dt'} = 1 - H \hat{\rho}(t') > 0.\]

(8)

This, in turn, implies that

\[H < \frac{1}{\hat{\rho}},\]

(9)

This is a necessary condition for developable extensibility of the base-curve. Indeed, we can see this another way.

The distance \(d\) that we have to go along \(\hat{d}(t')\) in order to move a distance \(h\) orthogonally from the base curve is similarly determined by trigonometry to be

\[d = h \sqrt{1 + \rho^2}.\]

(10)

Inserting this form into equation 6 for the principal curvature, one obtains

\[
\kappa_p(t, h) = \kappa \left[ \frac{1 + \rho^2}{1 - h \rho} \right](t'),
\]

(11)

which is simple enough to allow interpretation. One should remember that \(\kappa\) and \(\rho\) on the left-hand-side of this equation are evaluated at the point \(t'\) related to \(t\) by equation 7. The denominator of this expression vanishes when equation 9 for \(H\) becomes an equality. The principal-curvature then diverges implying that the extension has developed a cone-like vertex. It is now easily seen that large \(\rho\) and large positive \(\hat{\rho}\) affect the curvature unfavorably.

If \(H\) is small enough, equation 7 may be solved by iteration

\[t' = t + H \rho(t + H \rho(t + ...)).\]

(12)

To lowest order this yields

\[
\vec{r}(t, h) = \vec{r}(0) + h \hat{b}(t),
\]

(13)

which implies that the ribbon must lie mainly in the direction of the binormal.

When searching for an optimal winding path, initial conditions are given not just for the base-curve but for the entire strip, in that \(\vec{r}(0, h)\) is given for all \(h\) where \(0 < h < H\). On the other hand, this exact solution implies that the initial strip position is completely determined by the base-curve alone. As a result, it is possible that a selected path conflicts with the initial conditions. In general, enforcement of complete agreement between the two is a complex
nonlinear problem. For the windings considered here, \( p(t) \) vanishes at the start of the base-curve. As a result, the initial condition for the strip, i.e. that it lies in a constant \( \theta \) plane, is consistent with the base-curve provided that \( H \) is small enough.

III. - WINDING PATH OPTIMIZATION

If the cable is assumed to be mechanically homogeneous, then the bending energy of the cable can be shown to be proportional to the surface integral of the square of \( \kappa_p \).

One possible winding criterion is to minimize this bending energy. However, the cable in question is not actually a homogeneous Young's modulus structure. It is quite inhomogeneous and asymmetric in construction and behavior. Indeed, the cable is most likely to fail or delaminate at points where the local stress is high. As a result, a more practical and much simpler criterion is to select a path which minimizes the maximum local value of \( \kappa_p \) found along the strip. This min-max criterion is easily implemented on a computer. Only the values of \( \kappa_p \) at \((t, 0)\) and \((t, H)\) need to be considered since curvatures in the middle of the strip must lie between the extrema of these two values.

Equation 11 for \( \kappa_p \) implies that one should not only try to minimize the local curvature \( \kappa(t) \) but also restrict the torsion \( \gamma(t) \) to locations where \( \kappa \) is large so that their ratio \( \rho \) is small. Exhaustive laboratory experiments with a strip of paper have demonstrated the dramatic effect that torsion can have on the principal curvature if the longitudinal curvature \( \kappa \) is small.

The arclength \( t \) of the base-curve is a priori known only in the simplest cases. Instead, the base-curve is usually described in terms of a parameter \( p \) through a functional expansion of \( z(p) \) and \( \theta(p) \), for example

\[
\begin{align*}
\theta(p) &= \sum_{n=0}^{N} a_n p^n, \\
z(p) &= \sum_{n=0}^{N} b_n p^n.
\end{align*}
\]  

At \( p = 0 \) \((t = 0)\) we require that \( z = 0 \) and \( \theta \) be a given starting angle. There, \( d\theta/dp \) is zero while \( dz/dp \) is large. At the top of the mandrel, \( p = p_{\max} \), we require that \( \theta = \pi/2 \). There, \( dz/dp \) is zero while \( d\theta/dp \) is large.

\( dp/dt \) can be evaluated at \( p = p_{\max} \) in the limit that \( d\theta/dp \) goes to zero while \( dz/dp \) and \( d^2\theta/dp^2 \) remain finite, with result \( r_0 dp/dt = 3 \). Widths are technically limited, in this case, to \( H < r_0/3 \). However, this is not a fundamental limit and may be avoided by arranging that higher derivatives of \( \theta \) vanish at \( p = 0 \). In practice, this point of transition from a straight line into a curve can be brushed over in a computer code. A tiny amount of hard direction bending avoids the difficulty. Real troubles are not found near the start of the end winding, but only further along its course.

Our approach has used both Fourier and polynomial expansions. In both cases, coefficients of the expansion functions are varied by brute force and the maximum principal curvature along the strip sought for each combination of coefficients. Coefficients which lead to a violation of condition 9 are discarded. Combinations of coefficients yielding a min-max curvature are quickly found by this process despite its brute force nature. Figure 2 shows the this optimized min-max principal curvature as a function of starting angle for a ribbon, approximating the SSC dipole, whose width is one half of the mandrel radius. It is generally found that the min-max curvature increases with starting angle, as seen in this figure, independent of the details of parameterization. Windings which start nearest the top of the magnet will experience the highest stresses.

IV. - CONCLUSIONS

A method for the optimization of stress reduced magnet-end winding paths for ribbon-like superconducting cable has been presented. The method is applicable to winding mandrels of arbitrary geometry and may be useful in other areas of engineering where the stresses in thin sheets of rigid material are important. However, engineers must ultimately consider the non-ideal nature of the material being wound when applying this method.

References

