Stationary Longitudinal Phase Space Distributions with Space Charge

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Abstract

An iterative technique has been employed to find longitudinal phase space distributions which are stationary under the influence of a constant, imaginary Z/n. In the positive mass regime (where $\text{Im}(Z/n)$ is positive below transition or negative above transition), the main result is that, irrespective of the starting distribution, as the intensity is raised, the distribution tends towards one which has a parabolic line density. Since the locally elliptic distribution already has a parabolic projection, it is always stationary. In the negative mass regime, where the impedance augments the longitudinal focusing, centrally peaked stationary distributions cease to exist beyond a certain threshold intensity. In this regime, the limiting line density is also a parabola but one which is of opposite sign, i.e. with minimum density at the centre.

I. Introduction

In the study of beam stability, it is important to distinguish between stability and stationarity. Stability should only be defined with respect to a stationary distribution, because if a given distribution is perturbed in some way, it is not sensible to ask whether the perturbation grows unless the unperturbed distribution is stationary. With a short-range wake field such as space charge, the stationary distribution changes with beam intensity. This has recently been emphasized by Oide and Yokoya [1] who showed that mode-coupling calculations which ignore potential well distortion give different instability thresholds and growth rates than those which are self-consistent.

An additional source of confusion between stability and stationarity is the fact that the Keil-Schnell stability criterion, applied locally using the space charge impedance, is formally equivalent to the condition that the bucket area remain finite [2]. Hence, the Keil-Schnell criterion, applied with a predominantly reactive impedance to a bunched beam, pertains more to the stationarity of the distribution than to its stability.

We restrict ourselves to the space charge longitudinal wake given by an imaginary impedance proportional to frequency. It is well-known that then the space charge force is proportional to the derivative of the beam's line density. Hence, for the 'locally elliptic' distribution [2] (whose projection gives a parabolic line density), the space charge force is linear and, with a linear external focusing force, the phase space trajectories lengthen but remain elliptical. Analogously to the Kapchinsky-Vladimirsky distribution of the transverse case, this is the only distribution which does not change shape with intensity. As a result, tune shifts, bucket areas, and even longitudinal eigenmode frequencies [3] can be calculated by means of simple formulae. However, it is a highly idealized distribution, possessing no tails, and it is not difficult to see that an infinite number of other stationary distributions exist. In the following sections, we investigate three stationary distributions whose individual zero intensity line densities are, respectively, gaussian, hollow, and flat-topped.

II. Theory

This section presents Sacherer's theory [4] for transverse stationary distributions adapted to the longitudinal case. There are two requirements for a stationary distribution. (1) Particle motion must be derivable from a Hamiltonian. (2) There must be one or more integrals of motion with no explicit time dependence; i.e. there must be a relation $J(p,q)$=constant between a particle's momentum p and position q. The stationary distribution ($\rho(p,q)$) is then given by

$$\rho(p,q) = f(J(p,q)) \tag{1}$$

where f is an arbitrary function.

For longitudinal motion in synchrotrons, the effect of the accelerating cavities can be approximated as non-local, since the synchrotron frequency is in general much less than the revolution frequency. In that case, the Hamiltonian itself has no explicit time dependence and we can use H for J. Normalizing in such a way that the unperturbed phase space trajectories are circles, the space charge Hamiltonian is

$$H = \frac{p^2}{2} + \frac{q^2}{2} + I\lambda(q). \tag{2}$$

I is the intensity parameter, included explicitly here because the line density

$$\lambda(q) = \int f(H(p,q))dp \Big/ \iint f(H(p,q))dpdq \tag{3}$$

has been normalized to unity. We allow I to have either sign. A negative value corresponds to the 'negative mass' regime where the intensity-dependent part of the potential is attractive and there is potential-well bunch shortening. This can arise for space charge above transition or 'inductive-wall' impedance below transition.

We wish to find how the phase space trajectories (H=constant) and the line density deform as the intensity is raised for a given choice of f. The numerical method consists of an iteration where an old λ (as defined on a finite number of points q) is used in H to generate a new

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one by (3). In some cases, this procedure diverges and a relaxation technique (where a λ is generated from a weighted mean of old and new) is required.

One undesirable feature of the Hamiltonian (2) is that as I is changed, the average value of H changes as well, and this can change the nature of the distribution as we shall see below. For this reason, a constant is usually added to H and its value is changed along with I to keep some feature of the distribution constant. For example, adding $-I\lambda(0)$ makes invariant the slice $f(p,0)$.

III. Gaussian

The gaussian case, $f(H) = e^{-H}$, is singular in that the momentum dependence can be factored out:

$$\lambda(q) = \lambda(0)\exp(-q^2/2 - I[\lambda(q) - \lambda(0)]). \quad (4)$$

This is the Haissinski equation [5] applied to our particular case. For large positive I, in particular, $I \gg |\ln\lambda|/\lambda$, the line density becomes parabolic:

$$\lambda(q) = \lambda(0) - \frac{q^2}{2I}. \quad (5)$$

This can be seen on the right in Fig. 1[1]. It is analogous to space charge dominated transverse transport [7], where the distribution adjusts to cancel the external focusing force. The concept of Debye-shielding introduced in [7] applies here as well.

In the negative-mass or bunch-shortening regime, no solution is found beyond a certain intensity. This can be seen by Taylor-expanding λ in (4). For $\lambda(q) = \lambda(0)(1 + aq^2 + bq^4 + ...)$, we find

$$a = \frac{-1/2}{1 + I\lambda(0)}, \quad (6)$$

and clearly there is a singularity if $I = -\lambda(0)$. At this intensity, the phase space trajectories at the centre of the distribution are vertical. See the plot on the left in Fig. 1. Increasing the intensity further causes the iteration to quickly diverge with the momentum spread at the beam centre becoming infinite. This case is analogous in some respects to a black hole.

Not surprisingly, the condition $I > -\lambda(0)$, when converted back into unnormalized units, is exactly the same as the condition for avoiding the negative mass instability in a coasting beam with gaussian momentum distribution.

[1] The density plots were made with the DENSITY/DIFFUSION command in the graphics package PLOTDATA [6]. This uses a neighborhood halftoning process in which a pixel is turned on, or not, depending both upon the value of the density function at that point and upon a correction factor based on previous data values passed through an error filter. Signal responses are thus diffused over a weighted neighborhood. In this way, statistical fluctuations are avoided, but disadvantages include: correlated artifacts, directional hysteresis due to the raster order of processing, and transient behaviour near edges or boundaries. The last is the reason that the density plots shown here are not symmetric about the q-axis, but have a 'soft' edge in the upper half of the plane and a harder edge in the lower half.

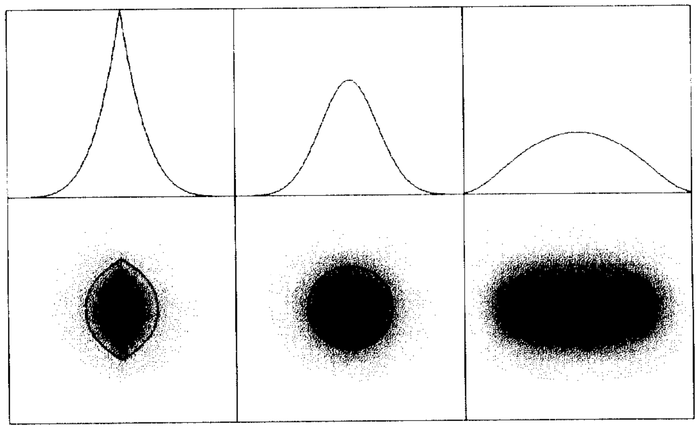

Figure 1: Line densities (top) and density plots (bottom) of stationary distributions for the gaussian case at intensities (left to right) $I = -1.552$, 0, and 20. The left-most plot is at threshold: for $I < -1.552$, no stationary distributions with $f(H) = e^{-H}$ exist. A few iso-density contours have also been plotted. On the contours, H=constant and, therefore, they also represent phase space trajectories. In these plots, as well as those following, the ranges of p and q are from -4 to 4.

IV. Hollow

We specify a hollow distribution as

$$f(H) = \exp[-(H - H_b)^2/2], \quad (7)$$

i.e. gaussian in H, *not* in the phase space coordinates. We set H_b=constant$-I\lambda(0)$ as described in section II. to maintain the hollowness as I is raised. The results with a choice of 2 as the constant are shown in Fig. 2 with a positive mass example on the right and a negative mass example on the left. In both cases we see the line density tending toward parabolic, but in the positive mass case an inverted cusp develops in the line density and no stationary distributions can be found for I greater than about 20. This is very similar to the threshold for the gaussian in the negative mass regime. It shows that a hole in the positive mass regime has a negative mass character.

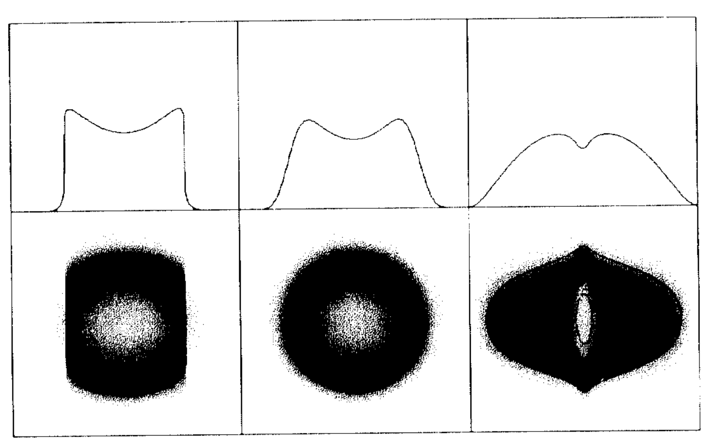

Figure 2: Line densities (top) and density plots (bottom) of stationary distributions for the hollow beam (7). Intensities (left to right) are $I = -8$, 0, and 20.

In the negative mass case, no threshold is found and the line density tends towards a parabola as $|I|$ is raised.

However, the parabola is inverted from the normal case and the slopes at the head and tail become very large. Particles in the bunch travelling towards the head or tail experience a very large space charge force at the edge and are reflected. This is evident from the outer contour (phase space trajectory) in the phase space density plot (Fig. 2, left). The length of the turn-around region is essentially a Debye screening length [7].

V. Flat-Topped

It was pointed out in [2] that a flat-topped line density can be realized by subtracting two locally elliptic distributions. The attraction is that for a given bunch population, flattening the top reduces the peak line density, thereby reducing the *transverse* tune shift.

Specifically, we define the density function as

$$f(H) = \sqrt{H_2 - H}^{*} - \sqrt{H_1 - H}^{*} \tag{8}$$

where $\sqrt{x}^{*}$ is $\sqrt{x}$ for $x > 0$ but 0 for $x < 0$. We set $H_1 = H_0 + I\lambda(0)$. Then with H_2 and H_0 fixed, the line density is independent of I. It is flat to $|q| = q_0 \equiv \sqrt{2H_0}$ and proportional to $H_2 - q^2/2$ from q_0 to $q_2 \equiv \sqrt{2H_2}$. As I is raised, we see in Fig. 3 that 'ears' develop in phase space. Continuing to raise I flattens the 'ears' till they collapse to zero area at

$$I = \frac{2}{3}(q_2^3 - q_0^3). \tag{9}$$

Above this threshold no stationary distributions exist. As a check, we can set H_0=0 and we recover the known threshold $I = \frac{2}{3}q_2^3$ above which no bucket exists for the locally elliptic distribution [2].

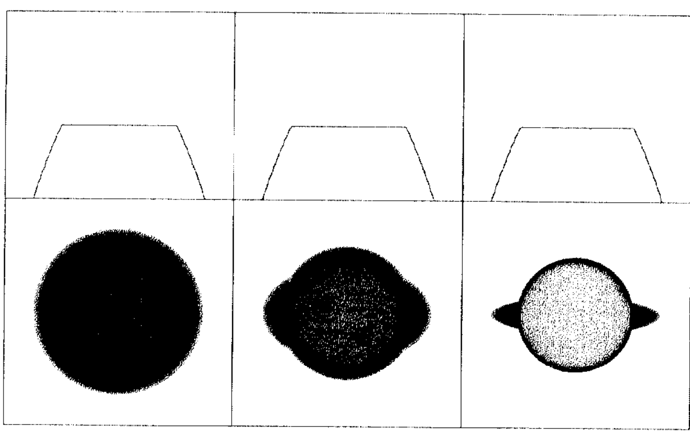

Figure 3: Line densities (top) and density plots (bottom) of stationary distributions for the flat-topped line density case (8) with H_2=4.5 (q_2=3) and H_0=2 (q_0=2). Intensities (left to right) are I = 0, 8, and 12. For the chosen parameters, we find from (9) that no stationary distributions with the indicated line density exist for $I > 38/3$.

From (9) we see that flattening a line density (raising q_0) reduces the longitudinal threshold. This must be balanced against the gain obtained in the transverse space charge limit.

VI. References

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