Scaling laws are relations between accelerator parameters (electric field, rf wavelength etc.) and beam parameters (current, energy, emittance, etc.) that define surfaces of constant accelerator performance in parameter space. These scaling laws can act as guides for designing radio-frequency quadrupoles (RFQs). We derive several scaling relations to show the various tradeoffs involved in choosing RFQ designs and to provide curves to help choose starting points in parameter space for optimizing an RFQ for a particular requirement. We show that there is a unique scaling curve, at a synchronous particle phase of $-90^\circ$, that relates the beam current, emittance, particle mass, and space-charge tune depression with the RFQ frequency and maximum vane-tip electric field, provided that we assume equipartitioning and equal longitudinal and transverse tune depressions. This scaling curve indicates the maximum performance limit one can expect at any point in any given RFQ. We show several examples for designing RFQs using this procedure.

**Introduction**

We define a procedure for obtaining initial RFQ designs. Scaling laws, derived below, are used to obtain an initial estimate of the RFQ parameter regime to satisfy the beam-dynamics requirements. RFQ optimization can then be done using program RFQDES, which is a general-purpose RFQ design program that allows maximum flexibility in choosing RFQ design algorithms.

We start by writing the linear space-charge force parameters for a uniform charge-density ellipsoid, then list the RFQ Mathieu equation parameters (the Mathieu equation approximately describes particle motion in the RFQ), and finally combine the RFQ and space-charge Mathieu terms to form scaling laws. Scaling-law plots, made to facilitate RFQ designs, are given, and we show several examples of RFQ designs using these laws.

**Space-Charge Forces**

Linear space-charge defocusing terms are calculated from the electric field components for a uniformly charged ellipse. The space-charge (Coulomb repulsion) defocusing terms (constant term in the Mathieu equation) for nonrelativistic beams are

$$\delta_L = J \Delta f / (R_M^2) \quad \text{(longitudinal)} \quad \text{and} \quad \delta_T = J \Delta f / (R_M^2) \quad \text{(transverse)},$$

where $J = 3 \pi I / (4 \gamma)$, and $f$ is a form factor that depends on $Z_M \sqrt{\psi} / R_M$; that is, $f = f(Z_M \sqrt{\psi} / R_M)$.

**RFQ Parameters**

The RFQ is a device that provides transverse focusing, longitudinal sinusoidal bunching, and acceleration of beam particles. The RFQ's electrical properties are determined by using an electrostatic potential function, which then gives the shape of the vanes and is used to calculate the electric fields for beam-dynamics modeling.

We define the following parameters:

- $k = 2 \pi / \lambda$.
- $A = (m^2 - 1)/(m^2 \gamma^2)$.
- $I = \text{beam current in amperes}$.
- $I_0 = \text{modified Bessel function}$.
- $\sigma_{T0}(Lo) = \text{transverse (longitudinal) phase advance per period (zero current)}$.
- $\sigma_T(L) = \sigma_{T0}(Lo) \sqrt{1 - \delta_T(L)} = \sigma_{T0}(Lo) / (1 + \delta_T(L))$, space-charge depressed.
- $\lambda = \text{free-space rf wavelength of the RFQ}$.
- $\psi = \text{beam particle's rest-mass energy in electron volts}$.

**Summary**

Scaling laws are relations between accelerator parameters (electric field, rf wavelength etc.) and beam parameters (current, energy, emittance, etc.) that define surfaces of constant accelerator performance in parameter space. These scaling laws can act as guides for designing radio-frequency quadrupoles (RFQs). We derive several scaling relations to show the various tradeoffs involved in choosing RFQ designs and to provide curves to help choose starting points in parameter space for optimizing an RFQ for a particular requirement. We show that there is a unique scaling curve, at a synchronous particle phase of $-90^\circ$, that relates the beam current, emittance, particle mass, and space-charge tune depression with the RFQ frequency and maximum vane-tip electric field, provided that we assume equipartitioning and equal longitudinal and transverse tune depressions. This scaling curve indicates the maximum performance limit one can expect at any point in any given RFQ. We show several examples for designing RFQs using this procedure.

**References**

1. K. A. Wadlinger, AT-2, MS H818.
2. Los Alamos National Laboratory, Los Alamos, NM 87545 USA.
4. SCALING LAWS FOR RFQ DESIGN PROCEDURES.
5. © 1985 IEEE. Personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution to servers or lists, or to reuse any copyrighted component of this work in other works must be obtained from the IEEE.

**Equations**

- $m = \text{RFQ modulation parameter} = (\text{maximum/minimum}) \text{vane radius}$.
- $\psi_L(L) = \text{transverse (longitudinal) space-charge tune depression parameter}$.
- $\psi_S = \text{synchronous phase (normally negative)}$.
- $\psi = \text{flutter factor} = (\text{maximum/minimum}) \text{beam radius}$.
- $R_M, R_M$ = maximum beam radius and $\delta r / ds$ (s is the distance along the trajectory/dx).
- $V = \text{RFQ vane voltage}$.
- $Z_M, Z_M = \text{maximum beam length} / 2$ and $\delta z / ds$ (s is the distance along the trajectory/dx).
- $\omega_0 = \text{impedance of free space} = 376.73 \Omega$.

**Below, all lengths are divided by the rf wavelength and all electric potentials and fields and the current are divided by the particle's rest-mass energy in electron volts. Consequently, all parameters that appear in the following equations are unitless and the equations do not explicitly contain $\lambda$ and $\psi_0$. We will return to SI units after we derive the scaling laws.
device may not occur on the vane tip and for smooth vanes is typically 1.4 larger than given here.)

We obtain the equations in this section the two equations (4) and (5) and the flutter factor $\psi$ (Eq. 6) used for the scaling laws

$$2a^2 + \gamma^2 c_0^2 = E_0^2 / (\gamma^2 + a^2) , \quad (4)$$

$$\gamma^2 c_0^2 = -\frac{2}{E_0} a \sin (\psi_a) / (\gamma^2 \sin \lambda) . \quad (5)$$

Finally, we present these equations in a form that facilitates making RFQ designs for given, desired beam parameters.

We define "emittance" as the area/n enclosed by an ellipse having its semimajor axis $R_M$, and semiminor axis $R_M$ (obtaining $c_T = R_M / R_M = \sigma_x / \sigma_y$, and

$\sigma_y = \sqrt{2} \sigma_x / \gamma^2$ and use the equipartitioning theorem* (which relates the free thermal energy in the longitudinal to the transverse direction) to combine these equations into $c_T = \sigma_x / \sigma_y$. We introduce a quantity $g$ defined by $g = J(c_T^2) / (\sigma_x^2 \sigma_y^2)$, which indicates the deviation from equipartitioning. We can now show that $R_M = \sqrt{c_T / \sigma_y}$, and $Z_M = \sqrt{(\sigma_x / \sigma_y)} / (\sigma_x)$. The form factor introduced above can now be written as

$$f(Z_M, \gamma^2 / R_M) = f(c_T / \sigma_y / (\sigma_x / \sigma_y)) \quad (7)$$

Combining Eqs. (1) and (2) and the equations in this section, we obtain two important relations

$$3g(1 - \psi_a)^{1/2} (1 - \psi_a)^{1/2} = c_0^2 / (\gamma^2 + \sigma_y^2) , \quad (8)$$

$$2f(1 - f) = \psi_a c_0^2 / (\psi_x c_0^2) . \quad (9)$$

We now study the RFQ Eqs. (4) and (5). We want to determine performance limits from the scaling laws. We therefore let $\psi_x = \gamma - 2$. The RFQ vane radius is related to the acceptance ($A_T$) by $a = \sqrt{A_T / \sigma_T}$. Let $b$ be the ratio of acceptance/emittance ($h = A_T / \psi_x$); then,

$$a = \sqrt{b} R_M = \sqrt{b} / \psi_x / \gamma^2 . \quad (10)$$

We can treat $[E_0(1 - \psi_T)^{1/4} / \psi_T] = \psi_x c_0^2 / (\psi_x c_0^2) (4e^2 \gamma^2 / \hbar^2)$. (11)

Combining equations we have

$$A = \left[ \frac{m^2 - 1}{m^2} \right] [m^2 \sigma_y^2 / \sigma_x^2] + \left[ \frac{m^2 \sigma_y^2 / \sigma_x^2} {\sigma_x^2} \right] \quad (12)$$

We revert to SI units in making this plot so that the scale will be more obvious. The units used are $E_0$ (volts/meter), $\gamma_T$ (meter-radians), $SI$ (amperes), $\sigma_0$ (elevtron volts). (The dimensionless parameters $E_0$, $\gamma_T$, and $\psi_x$ are the following functions of the dimensioned parameters:

$$E_0 = E_0 \psi_T \psi_0 / \sigma_0 , \quad \gamma_T = \psi_T / \sigma_0 , \quad I_T = I_T / \sigma_0 , \quad \chi = I_T / \sigma_0 . \quad (13)$$

It is remarkable that the two constraints of equipartitioning and $\psi_T = \psi_0$ have led to a scaling relation defined as a single curve. Given a set of requirements on particle type, beam current, emittance, and maximum acceptable space charge $u$, only the peak surface vane-tip electric field at quadrupole symmetry and rf wavelength remain to be adjusted (within the
Fig. 1. Scaling law curves ($\mu_L = \mu_T$).

Constraints of the scaling curve to determine the maximum performance capability of a given RFQ design. One other scaling relation is indicated in Fig. 1 when the acceptance $\alpha_T$ is not equal to the emittance $\epsilon_T$. We plot $L_3$ versus $L_4$ for $\alpha_T/\epsilon_T = 4$, where $L_3$ is divided by $\sqrt{\alpha_T/\epsilon_T}$. The minor difference in the curves indicates that we can design for $\alpha_T = \epsilon_T$, then multiply the resulting electric field by $\sqrt{\alpha_T/\epsilon_T}$.

Almost the same considerations apply to Case 2 as to Case 1 above, except that Eq. (17) now has a $\mu_L/\mu_T$ ratio dependence. We will therefore get a different but unique scaling curve for each different ratio of $\mu_L/\mu_T$. Two cases are shown in Fig. 2. For reference purposes in Figs. 1 and 2, we show curves of $\alpha_{00}$ and $\alpha_{01}$ versus $L_4$.

RFQ Design Examples and Discussion

Several observations concerning Fig. 1 also apply to Fig. 2 when the ratio $\mu_L/\mu_T$ is fixed and is constrained as in Eq. (14). First, any RFQ design that satisfies equipartitioning and $\mu_L - \mu_T$ will lie on the curve $L_3$ versus $L_4$. Second, suppose we wish to maximize beam current for a given zero current tune without regard to emittance. We can eliminate $\epsilon_T$ between $L_4$ and $L_3$ and solve for $S_1$ to obtain

$$S_1 = \frac{\lambda^3 (M_0 S_1^{1/3})}{\mu_L/\mu_T}.$$  

We can increase current for a given beam-dynamics characterization by increasing the rf wavelength. Note that with $L_3$ fixed, the emittance will increase linearly with current.

If the desired goal is to maximize the beam brightness for a given zero current tune, we can write

$$S_1 (\epsilon_T)^{1/3} = \frac{M_0^{1/3} \mu_L}{\epsilon_T} / \left[ \frac{E_0}{\epsilon_T} (1 - \mu_T)^{1/3} \right] = \lambda^{1/3}.$$  

In this case, the beam brightness will increase with increasing rf wavelength. For a given beam energy, power is proportional to current; therefore the increase in beam brightness will be accompanied by a proportional decrease in beam power. (Current and emittance are proportional to $\lambda$.)

As a simple example of using Fig. 1, suppose we wish to design a RFQ as an injector to an existing drift-tube linear accelerator. This existing device has an rf frequency of 425 MHz and accelerates a proton beam having 0.2-A current, $2 \times 10^{-6}$ $\pi$-rad transverse normalized emittance, and a maximum space charge $\mu$ of 0.7. We find

$$L_4 = 0.03 (1 - \mu_T)^{1/3} / \left[ \frac{E_0}{\epsilon_T} (1 - \mu_T)^{1/3} \right] = 0.0258.$$  

From Fig. 1 we find $L_3 = 445$. We then calculate $E_0$ using Eq. giving $4 \times 10^6$ V/m. This electric field is almost twice the Kilpatrick field limit and does not include any safety factors. If the beam-current requirement for the above case is 0.1 A, then the electric field determined from Fig. 1 is $13.7 \times 10^6$ V/m.

Conclusion

We have derived several scaling relations to show the various tradeoffs involved in choosing RFQ designs and have provided curves to help choose starting points in parameter space for optimizing an RFQ for a particular requirement. We have shown that there is a unique scaling curve that relates the beam current, emittance, and particle acceleration frequency and maximum vane-tip electric field and with space-charge tune depression—if we assume equipartitioning and equal longitudinal and transverse tune depressions. Finally, we have presented several examples for designing RFQs using our procedure.

Acknowledgments

I would like to thank Walter Lysenko for his encouragement and helpful suggestions concerning this subject matter.

References

1. Many of the equations that are stated in this paper can be found in T. P. Wangler, "Space-Charge Limits in Linear Accelerators," Los Alamos National Laboratory report LA-8388.