HAMILTONIAN FORMULATION FOR SYNCHROBETATRON RESONANCE DRIVEN BY DISPERSION IN RF CAVITIES

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Summary

Synchrobetatron resonance driven by dispersion in RF cavities is studied by using the Hamiltonian formalism. Explicit expressions are given for the growth rate of betatron and synchrotron oscillation amplitudes on resonance and on fast-crossing of the resonance. The effect of distributed RF cavities is also studied, and it is shown that the resonance can be suppressed by proper arrangement of RF cavities. The theory is shown to agree fairly well with the computer simulation for PETRA and some numerical examples are given for the booster of the TRIUMF Kaon Factory project. This paper is an abridged version of Ref. (1), and readers are referred to Ref. (1) for more details.

Hamiltonian

Starting from the well-known Hamiltonian for the single particle motion under electromagnetic fields and after several steps of canonical transformations, we obtain

$$H = \frac{B_0}{2} \kappa x^2 + \frac{1}{2} \beta \frac{p_x^2}{B_0} - \frac{R_n^2}{2} \omega_{rf}^2 \frac{p_x^2}{p_0^2} \frac{p_y^2}{p_0^2} + \frac{R_n^2}{2} \omega_{rf}^2 \frac{p_x^2}{p_0^2} \frac{p_y^2}{p_0^2}$$

$$- \omega_{rf}^2 \frac{p_x^2}{p_0^2} + \omega_{rf}^2 \frac{p_x^2}{p_0^2} + \omega_{rf}^2 \frac{p_x^2}{p_0^2}$$

$$+ \sin \left( \Delta \phi - \frac{1}{2} \omega_{rf}^2 \frac{p_x^2}{p_0^2} \frac{p_y^2}{p_0^2} \right).$$

The derivation of the Hamiltonian (1) is due to Corsten and Hagedoorn and is given in detail in the Appendix of Ref. (1). Here,

$$\xi = \alpha - \frac{1}{2} \beta \frac{B_n}{B_0}$$

$$\alpha = \text{momentum compaction factor}$$

$$\beta, \gamma = \text{Lorentz factors}$$

$$\kappa = \text{average radius}$$

$$p_0 = \text{central momentum}$$

$$x = \text{horizontal coordinate of betatron oscillations}$$

$$p_x = \text{canonical momentum conjugate to } x$$

$$\omega_{rf} = \text{RF angular frequency}$$

$$\omega = \text{transition energy}$$

$$\Delta \phi = \text{energy deviation from the synchronous value}$$

$$v_j = \text{RF voltage for cavity } j$$

$$\rho_0 = \text{sync. RF phase}$$

$$\Delta \phi = \text{RF phase relative to } \phi_0$$

$$D_j, D_j' = \text{dispersion and its derivative at cavity } j.$$

In the Hamiltonian (1), the angular position $\phi$ is used as an independent variable.

The Hamiltonian (1) can be split into two parts $H_0$ and $H_1$ by expanding the sinusoidal functions into Taylor series

$$H = H_0 + H_1,$$  \hspace{1cm} (2)

where $H_0$ contains the terms quadratic in canonical variables and gives the usual equations of synchrotron and betatron motions. $H_1$ denotes the perturbation which gives rise to synchrobetatron resonance. The perturbation Hamiltonian $H_1$ which gives rise to a synchrobetatron resonance $v = \frac{m}{n}$ where $m$ and $n$ are positive integers, is

$$H_1 = \frac{m}{2\pi} \omega_{rf} \frac{V_j}{V_0} \left( -\frac{D_{j}}{\rho_0} + \frac{D_{j}'}{\rho_0} \right).$$  \hspace{1cm} (3)

Here,$$

$$m = \begin{cases} m & (m = \text{even}) \\ m+1 & (m = \text{odd}) \end{cases}$$

Now we use action-angle variables $(I_x, \psi_x)$ and $(I_y, \psi_y)$ for synchrotron and betatron motion:

$$\Delta \phi = \frac{1}{2 \pi} \frac{2 - m}{m+1} \cos (\psi_x - \psi_y),$$  \hspace{1cm} (5)

$$\Delta \phi = \frac{1}{2 \pi} \frac{2 - m}{m+1} \cos (\psi_x + \psi_y),$$  \hspace{1cm} (6)

$$x = \frac{2 - m}{m+1} \cos (\psi_x + \psi_y),$$  \hspace{1cm} (7)

$$p_x = \frac{2 - m}{m+1} \left[ \alpha \cos (\psi_x + \psi_y) + \sin (\psi_x + \psi_y) \right].$$  \hspace{1cm} (8)

where $\psi_x$ and $\psi_y$ are synchrotron and betatron tunes, $\gamma_0$, $\beta_n$, are the magnetic fields of Courant and Snyder, $\delta$ is the phase angle of betatron oscillation given by $\frac{\delta}{\omega_{rf}} = h$, $h$ is the harmonic number and $\omega_0$ is the angular revolution frequency. The case below transition energy is assumed here. Above transition, $\omega_0$ in Eqs. (5) and (6) is replaced by $\omega_{rf}$.

Performing the transformations (5) $\psi \rightarrow \phi$ (8) in the Hamiltonian and knowing only slowly varying terms with $v = \frac{m}{n} - \frac{n}{n} \varepsilon$, where $e = \frac{\omega_0}{\omega_{rf}}$, we obtain

$$H_0 = \frac{v}{\alpha} - \frac{\beta}{\beta} \varepsilon = \frac{U}{\alpha} \varepsilon \left( \frac{U}{\alpha} \varepsilon \right),$$  \hspace{1cm} (9)

and

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\[ H_i = \frac{a}{2^m \cdot \pi} \left( \frac{2}{x^2} \right)^{\frac{1}{2}} \frac{1}{2^n} \cdot \left\{ D_{s,n} \sin(c+n \psi_{x,s}) + D_{c,n} \cos(c+n \psi_{x,s}) \right\} \]
\[ + \left( D_{c,n} \sin(c+n \psi_{x,s}) - D_{s,n} \cos(c+n \psi_{x,s}) \right) \]

where \( \psi \) is the maximum phase of synchrotron oscillations given by:
\[ \psi_m = \frac{2\pi^2 | \eta | \omega | f |}{c^2 | \omega |} \]

and
\[ \Gamma = \left( D_{s,n} \right)_{x} + \left( D_{c,n} \right)_{s} \]

\[ w_{c,n} = \left( E_{c,n} \right)_{j} \cos \left( \frac{n \psi_{x,j}}{x_{j}} \right) \]

\[ D_{s,n} = \left( E_{s,n} \right)_{j} \sin \left( \frac{n \psi_{x,j}}{x_{j}} \right) \]

\[ F_{c,n} = \left( E_{c,n} \right)_{j} \cos \left( \frac{n \psi_{x,j}}{x_{j}} \right) \]

\[ F_{s,n} = \left( E_{s,n} \right)_{j} \sin \left( \frac{n \psi_{x,j}}{x_{j}} \right) \]

the summation being done over one revolution. In the Hamiltonian (10), \( \phi \) is replaced by the smoothed variable \( \theta \).

\[ J \left( \psi_m \right) = m \]}

The maximum change \( \delta(\psi_m) \) in the beam size per revolution is
\[ \delta(\psi_m) = \sqrt{\delta(x \psi_s)} \]

Similarly, remembering that the energy spread is related to \( I \) and \( I_s \) as in Eq. (b), we obtain the maximum growth rate per revolution of energy spread on resonance as
\[ \delta(\Delta E / E)_{\text{max}} = \frac{1}{c^2 \left( \Delta E / E \right)_{\text{max}}} \frac{\left( E \right)_{\Delta E}}{\left( E \right)_{\text{max}}} \]

Now we consider the effect of fast-crossing of the resonance. First we obtain from the canonical equations of motion using the Hamiltonian (16)
\[ \psi = \frac{1}{2} \left( \Delta \psi \right)_{\Delta} + \psi_0 \]

where \( \Delta \psi \) and \( \Delta \psi_0 \) are the change of the tunes \( \psi \) and \( \psi_0 \) per revolution, and \( \Delta \psi_0 \) are constants. Here we neglect the perturbation Hamiltonian \( H_{\Delta} \).

Also, from the Hamiltonian (17), we obtain
\[ \psi = \frac{1}{2} \left( \Delta \psi \right)_{\Delta} + \psi_0 \]

where \( \psi_0 \) is another constant. The integral of Eq. (25) is given completely by using the Fresnel integral, if we assume the interval of \( \psi \) to be large. We assume for simplicity that the change of \( I_s \) per resonance crossing \( \Delta I \) is small and obtain

\[ \text{Dynamics} \]

First we note, from the Hamiltonian (17) and the canonical equations of motion, that a relation
\[ m \frac{I_s}{x} = \text{constant} \]

holds below transition energy for the resonance \( \nu \) to \( \nu_s = m \). Above transition energy, the role of sum and difference resonances is interchanged. Eq. (18) is a relation of Plivinski and Wrulich. Below transition energy, the amplitude growth is limited for a difference resonance, but \( I \) is usually much larger than \( I_s \) and the growth in the amplitude of betatron oscillations is important even for a difference resonance. For example, typical values of \( I \) and \( I_s \) are \( I \approx 1.5 \times 10^{-4} \text{ eV} \cdot \text{sec} \) (corresponding to \( 10^8 \text{ mm} \cdot \text{rad} \) normalized emittance) and \( I_s \approx 0.0063 \text{ eV} \cdot \text{sec} \) for the TRIUMF Kaon Factory project.

Now we calculate the maximum growth rate per revolution of the beam size \( \nu \), and the energy spread \( \Delta E / E \) on resonance. From the Hamiltonian (17).

\[ \gamma_s = - \frac{1}{2} \left( \Delta \psi \right)_{\Delta} + \psi_0 \]

where
\[ \gamma_0 \] is a constant phase and the prime denotes differentiation with respect to \( \psi \).

The maximum change \( \delta(\psi_{\text{max}}) \) in the beam size per revolution is
\[ \delta(\psi_{\text{max}}) = \frac{1}{c^2 \left( \Delta \psi \right)_{\Delta} + \psi_0} \]

Similarly, remembering that the energy spread is related to \( I \) as in Eq. (b), we obtain the maximum growth rate per revolution of energy spread on resonance as
\[ \delta(\Delta E / E)_{\text{max}} = \frac{1}{c^2 \left( \Delta E / E \right)_{\text{max}}} \frac{\left( E \right)_{\Delta E}}{\left( E \right)_{\text{max}}} \]

Now we consider the effect of fast-crossing of the resonance. First we obtain from the canonical equations of motion using the Hamiltonian (16)
\[ \psi = \frac{1}{2} \left( \Delta \psi \right)_{\Delta} + \psi_0 \]

where \( \Delta \psi \) and \( \Delta \psi_0 \) are the change of the tunes \( \psi \) and \( \psi_0 \) per revolution, and \( \Delta \psi_0 \) are constants. Here we neglect the perturbation Hamiltonian \( H_{\Delta} \).

Also, from the Hamiltonian (17), we obtain
\[ \psi = \frac{1}{2} \left( \Delta \psi \right)_{\Delta} + \psi_0 \]

where \( \psi_0 \) is another constant. The integral of Eq. (25) is given completely by using the Fresnel integral, if we assume the interval of \( \psi \) to be large. We assume for simplicity that the change of \( I_s \) per resonance crossing \( \Delta I \) is small and obtain
The change of betatron oscillation emittance $\Delta\varepsilon_x$ per resonance crossing is obtained from the invariant (18)

$$\Delta\varepsilon_x = \frac{|a_m|}{r_s} \phi_m \frac{\sqrt{\varepsilon_x}}{m \varepsilon_x} \left( \frac{1}{A} \right) \frac{1}{r_s} \frac{1}{m \phi \frac{\varepsilon_x}{\varepsilon}}. \tag{26}$$

The change of synchrotron tune $\nu$ is adiabatic, but the betatron tune $\nu_x$ changes rapidly due to space charge detuning. Adding the effect of resonance crossings incoherently for time $t$, we obtain

$$\Delta\varepsilon_x = \frac{|a_m|}{r_s} \phi_m \frac{\sqrt{\varepsilon_x}}{m \varepsilon_x} \left( \frac{1}{A} \right) \frac{1}{r_s} \frac{1}{m \phi \frac{\varepsilon_x}{\varepsilon}} \left( T_0 + \frac{t}{T_0} \right)^{m-1}, \tag{27}$$

where $T_0$ is the revolution frequency and $\Delta \varepsilon_x$ is the maximum detuning.

**Numerical Example**

In order to check the theory, we compare the present theory with a computer simulation for PETRA. The results are summarized in Tables I and II for 1σ (one standard deviation) beam size and 1σ energy spread, and for 6σ beam size and 6σ energy spread, respectively. The theoretical values are calculated by Eq. (21). We see from these tables that the agreement between the theory and the simulation is fairly good except for the $m = 5, 6$ beam size case.

<table>
<thead>
<tr>
<th>Table I</th>
<th>m (order of resonance)</th>
<th>simulation</th>
<th>theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19 usec</td>
<td>19 usec</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>360 usec</td>
<td>612 usec</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>34 msec</td>
<td>18 msec</td>
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</table>

<table>
<thead>
<tr>
<th>Table II</th>
<th>m (order of resonance)</th>
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<th>theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24 usec</td>
<td>21 usec</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>48 usec</td>
<td>113 usec</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>312 usec</td>
<td>540 usec</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.07 msec</td>
<td>5.81 msec</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.065 msec</td>
<td>48.0 msec</td>
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</tr>
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</table>

As is seen from the formulae in the previous sections, the effect of the synchrobetatron resonance is suppressed if the quantity $A$ defined in Eq. (20) is made zero. One method is, of course, to make the dispersion and its derivative at the cavity zero. Another method is to use the phase relation given in Eq. (20). If the RF cavities are placed symmetrically and the horizontal tune $\nu_x$ and accordingly $n$ is chosen properly, $A$ becomes zero even if the dispersion and its derivative at the cavity are not zero. The latter method is employed at TRIUMF. The RF cavities are placed in a three-fold symmetry (the machine superperiodicity is chosen to be six) and the tune is chosen to be $\nu_x = 5.25 (n = 5)$ which is not a multiple of three.

### Table III

<table>
<thead>
<tr>
<th>m</th>
<th>$\delta\varepsilon_x$ (m/turn)</th>
<th>$\Delta\varepsilon_x / \varepsilon_x$</th>
<th>$\Delta\varepsilon_x / \varepsilon_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3.5 \times 10^{-4}$</td>
<td>0.20</td>
<td>13.0</td>
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<tr>
<td>2</td>
<td>$1.1 \times 10^{-5}$</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>3</td>
<td>$4.3 \times 10^{-5}$</td>
<td>0.01</td>
<td>0.01</td>
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<td>4</td>
<td>$6.6 \times 10^{-7}$</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>5</td>
<td>$1.4 \times 10^{-6}$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td>$1.4 \times 10^{-8}$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>$2.0 \times 10^{-8}$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>8</td>
<td>$1.5 \times 10^{-10}$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

### Table IV

As is seen from the formulae in the previous sections, the effect of the synchrobetatron resonance is suppressed if the quantity $A$ defined in Eq. (20) is made zero. One method is, of course, to make the dispersion and its derivative at the cavity zero. Another method is to use the phase relation given in Eq. (20). If the RF cavities are placed symmetrically and the horizontal tune $\nu_x$ and accordingly $n$ is chosen properly, $A$ becomes zero even if the dispersion and its derivative at the cavity are not zero. The latter method is employed at TRIUMF. The RF cavities are placed in a three-fold symmetry (the machine superperiodicity is chosen to be six) and the tune is chosen to be $\nu_x = 5.25 (n = 5)$ which is not a multiple of three.

### Conclusion

The Hamiltonian formalism is developed for the synchrobetatron resonance driven by dispersion in RF cavities. The canonical perturbation theory using the Hamiltonian of Eq. (1) gives some useful formulae that can be easily used for numerical evaluation of the effect. The effect of distributed RF cavities is also studied. If RF cavities are placed symmetrically and the horizontal betatron tune $\nu_x$ is chosen properly, the synchrobetatron resonance can be suppressed. Though this fact has been known for some time, it is given a mathematical expression by the formalism of the present paper. The theory is also shown to agree with the computer simulation for PETRA.

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**References**

1. T. Suzuki, KEK Preprint 84-21 (1985) and also to be published in Part. Accel.