A FIRST ORDER SPACE CHARGE OPTION FOR TRANSOPTR

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Summary

TRANSOPTR1, a beam transport design code, incorporates automatic optimization of a beam transport system under general constraints. This optimization can be performed for either first order (with or without space charge effects) or second order calculations. Space charge effects are calculated in the x-y plane for continuous beams or in x-y-z space for bunched beams using a new approach. The method integrates a set of differential equations describing the evolution of the c beam matrix along the main trajectory where the c matrix represents the beam ellipsoid in phase space. A spatially uniform charge density approximation is used and emittance growth is assumed to be negligible. Changes in the optimum beam transport conditions of a system caused by space charge effects can be readily evaluated with this code. Some applications of the code are given.

Theory

Both TRANSORT and TRANSOPTR represent the beam by the 6 x 6 beam correlation matrix c which is related to any point X on the beam envelope by

\[ X = X_0 + cX \]

In first-order beam transport calculations, the effect of a transport element on the beam envelope coordinates can be represented by the linear transformation

\[ X(2) = R_{12}X(1) \]

where \( R_{12} \) is characteristic of the element. Similarly the c matrix undergoes the following transformation

\[ c(2) = R_{12}c(1)R_{12}^T \]

Such transport calculations only consider those forces which are linear, i.e., the force can be written as \( F = kx \) where \( x \) is the displacement from the beam center.

The space charge force, being the electrostatic interaction of the beam with itself, is not linear. However, the effect on the beam envelope can be approximated by a linear force given by \( qEx = qkx \) along the x-axis with similar expressions for the force along the y and z-axes.

For continuous beams with uniform space charge density \( p \), the longitudinal field \( E_z \) is neglected. The beam shape is elliptical in the transverse plane and is specified by the 2 x 2 space correlation matrix:

\[ \sigma_{12} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \]

In an x-y coordinate system where \( \sigma_{12} \) is in normal form, i.e., \( \sigma_{xy} = 0 \), the force constants are given by:

\[ k_x = \frac{c}{c_0} \frac{b}{a+b} \]

\[ k_y = \frac{c}{c_0} \frac{a}{a+b} \]

where \( a = \sigma_{xx} \), \( b = \sigma_{yy} \) are the semi-axes of the ellipse.

Similarly a beam bunch with uniform space charge density \( p \) is an ellipsoid specified by the 3 x 3 space correlation matrix:

\[ \sigma_{12} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \]

In a coordinate system where \( \sigma_{12} \) is in normal form the force constants are:

\[ k_x = \frac{c}{c_0} \frac{b}{a+b} \]

\[ k_y = \frac{c}{c_0} \frac{a}{a+b} \]

\[ k_z = \frac{c}{c_0} \frac{d}{d+e} \]

where \( a = \sigma_{xx} \), \( b = \sigma_{yy} \), \( c = \sigma_{zz} \) are the semi-axes of the ellipsoid.
where $k_x$, $k_y$, $k_z$ are given in equation (5). The evolution of the $\sigma$ matrix through the transport system can then be found by integrating equation (7). If $\phi_{D\theta}$ or $\phi_{D\phi}$ are not in normal form, then a transformation must be found to a rotated reference frame where they are normal. $F_{\text{sc}}$ can then be evaluated by (10) and transformed back to the original coordinate system.

Implementation in TRANSOPTR

TRANSOPTR performs first order and second order transport calculations with the usual matrix techniques for problems not involving space charge. However, it performs first order space charge transport calculations by numerically integrating equation (7). For reliable integration TRANSOPTR uses the routine STIFFZ from AELIB with Adam's Bashforth predictor-corrector method. The routine is an error-controlled variable step size integrator for systems of differential equations. The error-controlled variable step size capability results in fast integration of systems where space charge effects are small, while providing accurate integration where these effects are large.

Situations where $\phi_{D\theta}$ or $\phi_{D\phi}$ are not in normal form are handled by finding their eigenvalues and matrix of normalized eigenvectors. The eigenvalues are the squares of the semi-axes and the eigenvector matrix is the rotation matrix to transform $F_{\text{sc}}$ in the normal coordinate system back to the laboratory system.

The function $g(u,v)$ in equation (6) is numerically integrated with the routine AGAUSS from AELIB which uses an adaptive Gaussian method and a controlled accuracy of $10^{-4}$.

Examples

Figure 1 illustrates results from TRANSOPTR used for designing the 50 keV RFQ proton injector\textsuperscript{5} where the injector's output must be matched to the RFQ acceptance. Curve 1 shows the optimum match found by adjusting the solenoid fields when space charge is absent. Curve 2 shows the same transport system with 10 mA effective current (100 mA with 90% space charge neutralization) up to the end of the second solenoid and 100 mA in the last drift space. In curve 3 TRANSOPTR has optimized the solenoidal fields for a match with the same currents. Computation time for TRANSOPTR calculations with space charge is 2 to 3 times less than that for TRANSPORT with comparable accuracy.

Figure 2 shows TRANSOPTR calculations of growth in the bunch length for beam from RFQ\textsuperscript{7}. Curve 1 considers only the effects of energy spread and curve 2 includes space charge effects for a 600 keV, 75 mA, 270 MHz beam.

Conclusions

Linear space charge effects in first order beam transport have been included in TRANSOPTR. A new approach is used that integrates a differential equation describing the evolution of the $\sigma$ matrix. This technique is especially useful when a focusing force, the defocusing space charge force in this case, varies continuously along the reference trajectory. As a result, TRANSOPTR is now a convenient and reliable code for optimizing the design of beam transport systems where space charge effects are significant.
Appendix

The non-zero F matrix elements for some common beam transport elements are:

For a drift space, \( F(1,2) = 1 \)

\[
F(1,2) = 1 \\
F(3,4) = 1 \\
F(5,6) = (1 + \beta^2) \cdot 1
\]

For a quadrupole magnet, the same non-zero drift space elements and:

\[
F(2,1) = -K \\
F(4,3) = K
\]

where \( K = B_a/(a B_0) \), \( a \) is aperture radius, \( B_0 \) is magnetic induction at \( a \), \( B_a \) is beam's magnetic rigidity.

For a bending magnet with field index \( n \) and bend radius \( r \), the same non-zero drift space elements and:

\[
F(2,1) = -(1-n)/r^2 \\
F(2,6) = 1/r \\
F(4,3) = -n/r^2 \\
F(5,1) = -1/r
\]

For a solenoid with axial magnet induction \( B_2 \), the same non-zero drift space elements and:

\[
F(1,2) = K \\
F(2,1) = K^2 \\
F(2,4) = K \\
F(3,1) = -K \\
F(4,2) = -K \\
F(4,3) = -K^2
\]

where \( K = B_2/(B_0) \).

References