The kicker fields introduce correlations between evaluated numerically for a prototype high-energy non-overlapping synchrotron bands, explicit expressions for the signal suppression factors are derived and evaluated numerically for a prototype high-energy storage ring.

1. Introduction

A beam of particles undergoing stochastic cooling is characterized by collective properties generated by the interparticle interactions induced by the feedback loop. The kicker fields introduce correlations between particle arrival times and their phase-space coordinates. Such correlations are then propagated coherently within the beam by the collective motion of the particles.

Figure 1 below illustrates the self-consistency between the 'cooling' loop and the 'feedback through the beam' loop and demonstrates how it leads in general to modifications in the kicker field from its strength in the absence of beam feedback. Here $I_0(\omega)$ is the unperturbed Schottky current and $\lambda(\omega)$ the coherent modulation to the current arising from the kicker fields and propagated collectively by the beam to the pick-up. If the

$$\lambda_0(t) = \Phi(\omega) V(t)$$

beam is described by a system response operator $\Phi(\omega)$ such that $\lambda(\omega) = \Phi(\omega) V(\omega)$, we see immediately from Fig. 1 that the modulated voltage $V(\omega)$ at the kicker is related to the uncorrelated voltage $V_0(\omega) = G(\omega)I_0(\omega)$ as $\Phi(\omega)V(\omega) = V_0(\omega)$ where $\Phi(\omega) = [1 - G(\omega)D(\omega)]$. For systems invariant under arbitrary time translations, e.g. continuous coasting beams, the response operator $D(\omega)$ is just a multiplicative scalar function $D(\omega)$ so that the effect of feedback through the beam is a shielding or suppression of uncorrelated signals at a given frequency by a frequency dependent factor $\xi(\omega) = 1 - G(\omega)D(\omega)$, i.e. $V(\omega) = V_0(\omega)/\xi(\omega)$ similar to the dielectric screening of test charge fields in a plasma. An essential difference for bunched beams in storage rings is that the response is non-stationary, i.e. it is not invariant under arbitrary time translations but invariant only under discrete translations which are multiples of the revolution period in the ring. As we shall see below, this periodic non-stationarity leads to the coupling of modulated signals at all frequencies which are discrete translations of each other by multiples of the revolution frequency as follows:

$$G(\omega)\lambda(\omega) = \sum_{k=-\infty}^{\infty} D_k(\omega)\lambda(\omega + kw_0)$$

### Abstract

We develop a self-consistent Vlasov theoretic description of the dynamic distortion of incoherent Schottky signals from a bunched beam undergoing stochastic cooling, arising from beam feedback. In the situation of non-overlapping synchrotron bands, explicit expressions for the signal suppression factors are derived and evaluated numerically for a prototype high-energy storage ring.

2. Vlasov analysis

The longitudinal phase-space variables of the particles are taken to be the action-angle variables $(a,\theta)$ of synchrotron oscillations. In order to incorporate synchrotron frequency spread which leads to the all important 'mixing' in phase space and favors the Landau damping, we represent the synchrotron oscillations by the quasi-linear orbit as $\theta = a \sin \omega t + a \sin \phi(t)$ where $\phi(t) = \phi_0(t) + Q(t)$ and $\phi(t) = \phi_0(t)$. To this order of approximation, the action variable is simply $J = (a^2/2w_0)$. We assume the existence of a zero-order stationary distribution function for the beam $f_0 = f_0(a) = g_0(J)$; $(df_0/da) = 0$. The propagation of small longitudinal perturbations $f_1(a,\phi; t)$ in phase-space is governed by the Vlasov equation, linearized in $f_1$, in terms of the amplitude-phase variables $(a,\phi)$ as follows ($\omega = f_0 + f_1$):

$$\frac{df_1(a,\phi; t)}{dt} + i\theta(a) \frac{df_1(a,\phi; t)}{d\phi} + a \frac{df_0(a)}{da} = \eta$$

The perturbed amplitude equation is given by,

$$\dot{a} = (q \omega^2/2m) s(t) \cos \phi(t)$$

where

$$s(t) = \omega_3 \sum_{n=0}^{\infty} V_{\text{en}}(\theta_n - \theta_0 - 2\pi n)$$

is the voltage sampled by the particle as a function of time as it passes through the kicker with voltage $V(t)$ and $\kappa = [d\omega(E)/dE]$ is the machine parameter. Fourier
expanding the periodic delta function and using the identity
\[ \cos y \exp (i x \sin y) = \sum_{n=-\infty}^{\infty} (\nu/x) J_n(x) \exp (i n y) \]
one obtains
\[ \hat{a} = \left(\frac{g(f(x))}{\omega(a)} \right) a \sum_{n=-\infty}^{\infty} \frac{J_n(x) e^{-in\theta_k[\nu(t) e^{i\omega x t}]}}{\sqrt{n}} e^{-i u y(t)} \]
where \( J_n(x) \) is an ordinary Bessel function and \( \nu = 0 \) is excluded from the sums. Substituting Eq. (10) in Eq. (6), Fourier series expanding in the periodic angle variable \( \psi \), Fourier transforming to frequency \( \Omega \) and using \( V(n) = \frac{\partial}{\partial \omega} G(\Omega) \) where \( G(\Omega) \) is the overall transfer function of the cooling loop, one obtains
\[ \hat{p}(a; \Omega) = \left(\frac{g(f(x))}{\omega(a)} \right) \left[ \frac{d^2}{d \psi^2} \right] \]
\[ \times \sum_{n=-\infty}^{\infty} \frac{J_n(x) e^{-in\theta_k[\nu(t) e^{i\omega x t}]}}{\sqrt{n}} e^{-i u y(t)} \]
where
\[ \exp (i x \sin y) = \sum_{n=-\infty}^{\infty} J_n(x) \exp (i n y) \]
are used to express \( \hat{a} \) in terms of \( a \) as:
\[ \hat{a} = \left(\frac{g(f(x))}{\omega(a)} \right) a \sum_{n=-\infty}^{\infty} \frac{J_n(x) e^{-in\theta_k[\nu(t) e^{i\omega x t}]}}{\sqrt{n}} e^{-i u y(t)} \]
Substituting Eq. (12) in Eq. (11), we obtain
\[ \hat{p}(J; \Omega) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int d\Omega' dJ' \hat{p}^a_{k}(J, J'; \Omega) \hat{p}(J'; \Omega) + \hat{p}(J; \Omega) \]
where
\[ \hat{p}^a_{k}(J, J'; \Omega) = \left(\frac{-i g(f(x))}{2 \pi} \right) \left[ \frac{d^2}{d \psi^2} \right] e^{-i k \psi} \]
\[ \times \sum_{n=-\infty}^{\infty} \frac{J_n(x) e^{-in\theta_k[\nu(t) e^{i\omega x t}]}}{\sqrt{n}} e^{-i u y(t)} \]
We note that Eq. (14) is consistent with a linear and causal propagator description of the response as
\[ \hat{C}(J, \psi, \Omega) = \int d\Omega' dJ' \hat{a} \hat{p}^a_{k}(J, J'; \Omega) \hat{p}(J'; \Omega) + \hat{p}(J; \Omega) \]
\[ \times e^{i \omega x t} \]
where \( e^{i \omega x t} \) is an arbitrary excitation at time \( t \) (incoherent Schottky signal excitation) and \( e(t, \Gamma', t, t') \) is the periodic non-stationary response kernel
\[ s(T, \Gamma', |t, t') = \sum_{k=-\infty}^{\infty} s_k(T, \Gamma', t-t') e^{i \omega x t} \]
where $f_{0}(a)$ is the amplitude distribution normalized to unity. For the particular case of a water-bag distribution $f_{0}(a) = (-1/A)q(1/a) - (1/A)$ where $q$ is the Heavyside step function, an expression for signal suppression is obtained without neglecting the principal value integral and one verifies

$$
\varepsilon_{s}(a) = 1 + \frac{\omega_{0}^{2} / (2\pi)}{2^{1} + \sum_{n} \frac{n^{2}}{n} \left( \omega_{0}^{2} + \omega_{n}^{2}(a) \right) \cos(\theta - \theta_{k})}
$$

\hspace{1cm} (27)

We note from Eqs. (26) and (27) that for linear synchrotron oscillations with $|d\omega_{s}(a)/da| = 0$ and $\omega_{0}(a) = \omega_{n}(a)$, signals get totally suppressed by beam feedback $f_{s}(a) = 0$. There is no mixing in phase-space and particles neighbouring in phase-space stay together coherently for arbitrarily large number of synchrotron oscillations, leading to persistent correlations. Moreover, signals from particles at the edge of a hard-edge distribution in amplitude get screened for non-zero mixing $(d\omega_{s}/da \neq 0)$. For finite steepness $(d\omega_{s}/da)$ and non-zero spread $(d\omega_{s}/da)$ in the bulk of phase-space, $\varepsilon_{s}(a)$ remains a finite number leading to non-zero suppressed signals which could be used for stochastic cooling, no matter how slow. We note that for longitudinal cooling the gain $G_{n}^{0} / (\alpha_{n})^{2}$ is such as to have opposite phases at $\omega_{0}(a)$ for the same $n$ in order to induce longitudinal cooling. A similar analysis of the transverse response yields for the transverse signal suppression

$$
\varepsilon_{n}(a) = 1 + \frac{\omega_{0}^{2} / (2\pi)}{2^{1} + \sum_{n} \frac{n^{2}}{n} \left( \omega_{0}^{2} + \omega_{n}^{2}(a) \right) \cos(\theta - \theta_{k})}
$$

\hspace{1cm} (28)

where $\Omega_{n}(a) = (n + Q)\omega_{0} + \omega_{n}(a)$, $Q$ the betatron tune, $\alpha$ the machine chromaticity, $n$ the off-energy function.

5. Numerical results and discussion

The transverse and longitudinal signal suppression factors for non-overlapping synchrotron bands are evaluated for different synchrotron harmonics $n$ and at different amplitudes for a bunch of $N = 10^{11}$ particles confined by $\sigma = 7776$ harmonic RF cavity with central revolution frequency $f_{c} = 50$ kHz in a storage ring with betatron tune $Q = 19.4$, small amplitude synchrotron oscillation time-period of 5 milli-second and for a cooling loop with a flat gain between 2 and 4 GHz. The particle distribution is taken to be $f(a) = (3/2a_{0}^{3}) \times [1 + \omega_{0}(a)^{2}]$, an analytic function with a sharp edge at $a_{0}$, leading to a parabolic line density. The synchrotron non-linearity is parametrized by $\omega_{0}(a) = \omega_{0}(0)[1 - (h^{2}a^{2}/16)]$. The maximum amplitude of particles in the bunch is taken to be 0.0012 radians slightly less than the maximum angular extent of the bucket given by $a_{\text{max}} = 0.00141$ radians. The gain $G$ is chosen so as to give an optimum cooling time of 60 hours for the highest amplitude particles in the bunch. The results are shown in Fig. 2 (a) and (b). Suppression is enhanced for low synchrotron harmonics in general and increases towards the core of the bunch before ultimately reducing to the value 1 at small amplitudes determined by $n$. Higher harmonics $n$ contribute to larger amplitudes only but with strengths less than those of the low harmonics. At any given amplitude only a finite number of $n$'s contribute. At smaller amplitudes less number of synchrotron modes contribute but with enhanced strengths. The longitudinal suppression at any given amplitude does not increase as $1/|\mu|$ as in the transverse case but rather levels to a flat value for almost all amplitudes as $|\mu| \rightarrow 0$.

![Fig. 2](image_url)

There is usually considerable amount of synchrotron band overlap in a high frequency large bandwidth cooling feedback system for beams with finite synchrotron frequency spread. In such overlap situation inverse of the operator $E(n)$ in Eqs. (2) and (3) is in general difficult to obtain. For lack of any general solution at present, one possible conjecture is that for sufficient overlap of synchrotron bands at high frequencies, the bunch behaves like a coating beam with the bunched structure manifesting in an enhanced effective gain given by the correlated orbit strengths $C_{n}^{(m)} / (\alpha_{n})^{2} \int f(a) \cos(\theta - \theta_{k})$. The correlation dominates over a range in $\mu$ which is the bunching factor $(\mu \mu_{0})$ where $\mu_{0}$ is the bunch duration. We may thus use an equivalent coating beam signal suppression expression, with the total number of particles $N$ modified to $\mu_{\text{eff}} = \mu N(\mu_{0})$, as discussed in accompanying papers.

References

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