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QUASICLASSICAL CALCULATIONS OF WIGNER FUNCTIONS IN NONLINEAR BEAM DYNAMICS

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Abstract

We present the application of variational-wavelet analysis to numerical/analytical calculations of Wigner functions in (nonlinear) quasiclassical beam dynamics problems. (Naive) deformation quantization and multiresolution representations are the key points. We construct the representation via multiscale expansions in generalized coherent states or high-localized nonlinear eigenmodes in the base of compactly supported wavelets and wavelet packets.

1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique based on local nonlinear harmonic analysis (wavelet analysis, generalized coherent states analysis) to quantum/quasiclassical (nonlinear) beam/accelerator physics calculations. The reason for this treatment is that recently a number of problems appeared in which one needs take into account quantum properties of particles/beams. Our starting point is the general point of view of deformation quantization approach at least on naive Moyal/Weyl/Wigner level [1], [2].

The main point is that the algebras of quantum observables are the deformations of commutative algebras of classical observables (functions) [1]. So, if we have the Poisson manifold M (symplectic manifolds, Lie coalgebras, etc) as a model for classical dynamics then for quantum calculations we need to find an associative (but non-commutative) star product $*$ on the space of formal power series in $\hbar$ with coefficients in the space of smooth functions on M such that

$$f * g = fg + \hbar\{f,g\} + \sum_{n\geq 2}\hbar^n B_n(f,g), \tag{1}$$

where $\{f,g\}$ is the Poisson brackets, B_n are bidifferential operators. Kontsevich gave the solution to this deformation problem in terms of formal power series via sum over graphs and proved that for every Poisson manifold M there is a canonically defined gauge equivalence class of star-products on M. Also there are nonperturbative corrections to power series representation for $*$ [1]. In naive calculations we may use simple formal rule:

$$* \quad\equiv\quad \exp\left(\frac{i\hbar}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)\right) \tag{2}$$

In this paper we consider calculations of Wigner functions (WF) as the solution of Wigner equations [2] (part 2):

$$i\hbar\frac{\partial}{\partial t}W(x,p,t) = H * W(x,p,t) - W(x,p,t) * H \tag{3}$$

and especially stationary Wigner equations. Our approach is based on extension of our variational-wavelet approach [3]-[14]. Wavelet analysis is some set of mathematical methods, which gives us the possibility to work with well-localized bases in functional spaces and gives maximum sparse forms for the general type of operators (differential, integral, pseudodifferential) in such bases. These bases are natural generalization of standard coherent, squeezed, thermal squeezed states [2], which correspond to quadratical systems (pure linear dynamics) with Gaussian Wigner functions. So, we try to calculate quantum corrections to classical dynamics described by polynomial nonlinear Hamiltonians such as orbital motion in storage rings, orbital dynamics in general multipolar fields etc. from papers [3]-[14]. The common point for classical/quantum calculations is that any solution, which comes from full multiresolution expansion in all space/time (or phase space) scales, is represented via expansion into a slow part and fast oscillating parts (part 3). So, we may move from the coarse scales of resolution to the finest one to obtain more detailed information about our dynamical classical/quantum process. In this way we give contribution to our full solution from each scale of resolution. The same is correct for contributions to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution. Because affine group of translations and dilations (or more general group, which acts on the space of solutions) is inside the approach (in wavelet case), this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole underlying infinite scale of spaces. In part 4 we consider numerical modelling of Wigner functions, which explicitly demonstrates quantum interference of generalized coherent states.

2 WIGNER EQUATIONS

According to Weyl transform, quantum state (wave function or density operator) corresponds to Wigner function, which is the analog of classical phase-space distribution [2]. We consider the following form of differential equations for time-dependent WF, $W = W(p,q,t)$:

$$W_t = \frac{2}{\hbar}\sin\left[\frac{\hbar}{2}(\partial_q^H\partial_p^W - \partial_p^H\partial_q^W)\right]\cdot HW \tag{4}$$

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Let

$$\hat{\rho} = |\Psi_\epsilon >< \Psi_\epsilon|$$

be the density operator or projection operator corresponding to the energy eigenstate $|\Psi_\epsilon >$ with energy eigenvalue ϵ. Then time-independent Schroedinger equation corresponding to Hamiltonian

$$\hat{H}(\hat{p},\hat{q}) = \frac{\hat{p}^2}{2m} + U(\hat{q}), \tag{5}$$

where $U(\hat{q})$ is an arbitrary polynomial function (related beam dynamics models considered in [3]-[14]) on $\hat{q}$, is [2]:

$$\hat{H}\hat{\rho} = \epsilon\hat{\rho} \tag{6}$$

After Weyl-Wigner mapping we get the following equation on WF in c-numbers:

$$H\Big(p + \frac{\hbar}{2i}\frac{\partial}{\partial q}, q - \frac{\hbar}{2i}\frac{\partial}{\partial p}\Big)W(p,q) = \epsilon W(p,q) \tag{7}$$

or

$$\Big(\frac{p^2}{2m} + \frac{\hbar}{2i}\frac{p}{m}\frac{\partial}{\partial q} - \frac{\hbar^2}{8m}\frac{\partial^2}{\partial q^2}\Big)W(p,q) +$$
$$U\Big(q - \frac{\hbar}{2i}\frac{\partial}{\partial p}\Big)W(p,q) = \epsilon W(p,q)$$

After expanding the potential U into the Taylor series we have two real partial differential equations. In the next section we consider variation-wavelet approach for the solution of these equations for the case of an arbitrary polynomial $U(q)$, which corresponds to a finite number of terms in equations (7) up to any finite order of $\hbar$.

3 VARIATIONAL MULTISCALE REPRESENTATION

Let L be an arbitrary (non)linear differential operator with matrix dimension d, which acts on some set of functions $\Psi \equiv \Psi(x,y) = \Big(\Psi^1(x,y),\dots,\Psi^d(x,y)\Big), \quad x,y \in \Omega \subset R^2$ from $L^2(\Omega)$:

$$L\Psi \equiv L(Q,x,y)\Psi(x,y) = 0, \tag{8}$$

where $Q \equiv Q(x,y,\partial/\partial x,\partial/\partial y)$.

Let us consider now the N mode approximation for solution as the following ansatz (in the same way we may consider different ansatzes):

$$\Psi^N(x,y) = \sum_{r,s=1}^N a_{rs}\Psi_r(x)\Phi_s(y) \tag{9}$$

We shall determine the coefficients of expansion from the following variational conditions (different related variational approaches are considered in [3]-[14]):

$$\ell^N_{k\ell} \equiv \int (L\Psi^N)\Psi_k(x)\Phi_\ell(y)\mathrm{d}x\mathrm{d}y = 0 \tag{10}$$

So, we have exactly dN^2 algebraical equations for dN^2 unknowns a_{rs}. But in the case of equations for WF (7) we have overdetermined system of equations: $2N^2$ equations for N^2 unknowns a_{rs} (in this case $d = 1$). In this paper we consider non-standard method for resolving this problem, which is based on biorthogonal wavelet expansion. So, instead of expansion (9) we consider the following one:

$$\Psi^N(x,y) = \sum_{r,s=1}^N a_{r,s}\Psi_r(x)\Psi_s(y) + \sum_{i,j=1}^N \widetilde{a}_{ij}\widetilde{\Psi}_i(x)\widetilde{\Phi}_j(y)$$

where $\widetilde{\Psi}_i(x), \widetilde{\Phi}_j(y)$ are the bases dual to initial ones.

Because wavelet functions are the generalization of coherent states we consider an expansion on this overcomplete set of basis wavelet functions as a generalization of standard coherent states expansion. So, variational approach reduced the initial problem (8) to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage.

We consider the multiresolution expansion as the second main part of our construction. We have contribution to final result from each scale of resolution from the whole infinite scale of increasing closed subspaces V_j:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$$

The solution is parametrized by solutions of two reduced algebraical problems, one is linear or nonlinear (10) (depends on the structure of operator L) and the others are some linear problems related to computation of coefficients of algebraic equations (10). These coefficients can be found by some wavelet methods. We use compactly supported wavelet basis functions for expansions (9). We may consider different types of wavelets including general wavelet packets.

Now we concentrate on the last additional problem, that comes from overdeterminity of equations (7), which demands to consider biorthogonal wavelet expansions. It leads to equal number of equations and unknowns in reduced algebraical system of equations (10). We start with two hierarchical sequences of approximations spaces:

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots,$$
$$\dots \widetilde{V}_{-2} \subset \widetilde{V}_{-1} \subset \widetilde{V}_0 \subset \widetilde{V}_1 \subset \widetilde{V}_2 \dots,$$

and as usually, W_0 is complement to V_0 in V_1, but now not necessarily orthogonal complement. Functions $\varphi, \tilde{\varphi}$ generate a multiresolution analysis. $\varphi(x-k)$, $\psi(x-k)$ are synthesis functions, $\tilde{\varphi}(x-\ell)$, $\tilde{\psi}(x-\ell)$ are analysis functions. Synthesis functions are biorthogonal to analysis functions. Scaling spaces are orthogonal to dual wavelet spaces. Biorthogonal point of view is more flexible and stable under the action of large class of operators while orthogonal (one scale for multiresolution) is fragile, all computations are much more simple and we accelerate the rate of convergence of our expansions (9). By analogous anzatzes and approaches we may construct also the multiscale/multiresolution representations for solution of time dependent Wigner equation (4) [14].

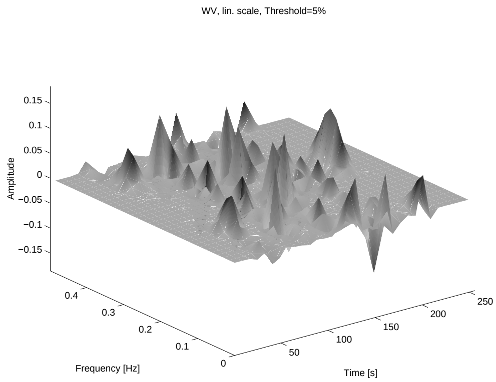

Figure 1: Wigner function for 3 wavelet packets.

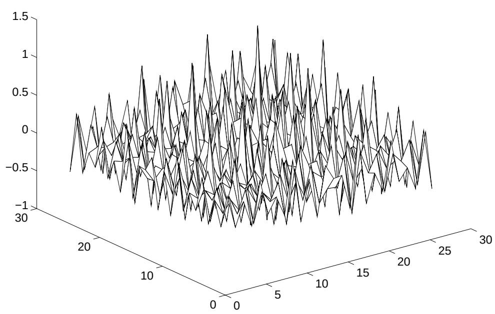

Figure 2: Multiresolution/multiscale representation for Wigner function.

4 NUMERICAL MODELLING

So, our constructions give us the following N-mode representation for solution of Wigner equations (7):

$$W^N(p,q)=\sum_{r,s=1}^{N} a_{rs}\Psi_r(p)\Phi_s(q) \qquad (11)$$

where $\Psi_r(p)$, $\Phi_s(q)$ may be represented by some family of (nonlinear) eigenmodes with the corresponding multiresolution/multiscale representation in the high-localized wavelet bases:

$$\Psi_k(p) = \Psi^{M_1}_{k,slow}(p)+\sum_{i\geq M_1}\Psi^i_k(\omega^1_i p), \quad \omega^1_i\sim 2^i$$

$$\Phi_k(q) = \Phi^{M_2}_{k,slow}(q)+\sum_{j\geq M_2}\Phi^j_k(\omega^2_j q), \quad \omega^2_j\sim 2^j$$

Our (nonlinear) eigenmodes are more realistic for the modelling of nonlinear classical/quantum dynamical process than the corresponding linear gaussian-like coherent states. Here we mention only the best convergence properties of expansions based on wavelet packets, which realize the so called minimal Shannon entropy property. On Fig. 1 we present the numerical modelling [15] of Wigner function for a simple model of beam motion, which explicitly demonstrates quantum interference property. On Fig. 2 we present the multiscale/multiresolution representation (11) for solution of Wigner equation.

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6 REFERENCES

[1] D. Sternheimer, Los Alamos preprint: math.QA/9809056, M. Kontsevich, q-alg/9709040, V. Periwal, hep-th/0006001.

[2] T. Curtright, T. Uematsu, C. Zachos, hep-th/0011137, M. Huq, e.a., *Phys. Rev.*, **A 57**, 3188 (1998).

[3] A.N. Fedorova and M.G. Zeitlin, *Math. and Comp. in Simulation*, **46**, 527, 1998.

[4] A.N. Fedorova and M.G. Zeitlin, *New Applications of Nonlinear and Chaotic Dynamics in Mechanics*, 31, 101 Kluwer, 1998.

[5] A.N. Fedorova and M.G. Zeitlin, **CP405**, 87, American Institute of Physics, 1997. Los Alamos preprint, physics/-9710035.

[6] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, Proc. PAC97 **2**, 1502, 1505, 1508, APS/IEEE, 1998.

[7] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, Proc. EPAC98, 930, 933, Institute of Physics, 1998.

[8] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, **CP468**, 48, American Institute of Physics, 1999. Los Alamos preprint, physics/990262.

[9] A.N. Fedorova, M.G. Zeitlin and Z. Parsa, **CP468**, 69, American Institute of Physics, 1999. Los Alamos preprint, physics/990263.

[10] A.N. Fedorova and M.G. Zeitlin, Proc. PAC99, 1614, 1617, 1620, 2900, 2903, 2906, 2909, 2912, APS/IEEE, New York, 1999.
Los Alamos preprints: physics/9904039, 9904040, 9904041, 9904042, 9904043, 9904045, 9904046, 9904047.

[11] A.N. Fedorova and M.G. Zeitlin, The Physics of High Brightness Beams, 235, World Scientific, 2000. Los Alamos preprint: physics/0003095.

[12] A.N. Fedorova and M.G. Zeitlin, Proc. EPAC00, 415, 872, 1101, 1190, 1339, 2325, Austrian Acad.Sci., 2000.
Los Alamos preprints: physics/0008045, 0008046, 0008047, 0008048, 0008049, 0008050.

[13] A.N. Fedorova, M.G. Zeitlin, Proc. 20 International Linac Conf., 300, 303, SLAC, Stanford, 2000. Los Alamos preprints: physics/0008043, 0008200.

[14] A.N. Fedorova, M.G. Zeitlin, Los Alamos preprints: physics/0101006, 0101007 and World Scientific, in press.

[15] F. Auger, e.a., Time-frequency Toolbox, CNRS/Rice Univ. (1996).