Fokker-Planck analysis of transverse collective instabilities in electron storage rings

Ryan Lindberg
Argonne National Lab

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Introduction

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- Emission of synchrotron radiation affects the instability threshold
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  - Leads to energy loss (damping) in all planes and diffusion of trajectories in phase space, so that the electron distribution function obeys a Fokker-Planck equation
  - These dissipative effects typically dominate those associated with Landau damping in high energy electron storage rings
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- First analysis of single bunch instabilities using the Fokker-Planck equation was made by T. Suzuki†, who focused on the zero-chromaticity limit and the traditional transverse mode coupling instability (TMCI)
  - Found that the Fokker-Planck dynamics implies that higher-order modes are damped more strongly
  - Since TMCI describes the merger of two low-order modes, the Fokker-Planck analysis makes a relatively small effect on the predicted instability threshold when $\xi = 0$
  - At large chromaticity we find that stability is dictated by high order modes, and the damping and diffusion of the Fokker-Planck equation increases the predicted stable current

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  - At large chromaticity we find that stability is dictated by high order modes, and the damping and diffusion of the Fokker-Planck equation increases the predicted stable current
- We have simplified Suzuki’s results, and applied them to “large” chromaticity

Overview of the Fokker-Planck analysis

\[
\frac{\partial F}{\partial s} + \{F, H\} = \frac{2}{c \tau_z} \left[ \sigma_0^2 \frac{\partial^2 F}{\partial p_z^2} + p_z \frac{\partial F}{\partial p_z} + F \right] + \frac{2}{c \tau_x} \left[ \varepsilon_0 \frac{\partial^2 F}{\partial J^2} + \frac{\varepsilon_0}{4J} \frac{\partial^2 F}{\partial \Psi^2} + (\varepsilon_0 + J) \frac{\partial F}{\partial J} + F \right]
\]
Overview of the Fokker-Planck analysis

Hamiltonian part:
linear (synchrotron + betatron) motion,
chromatic nonlinearity, and transverse wakefields

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Dissipative (Fokker-Planck) part:
Damping and diffusion due to the stochastic emission of synchrotron radiation
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Dissipative (Fokker-Planck) part:
Damping and diffusion due to the stochastic emission of synchrotron radiation

Longitudinal damping time
Energy spread
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- Longitudinal damping time
- Energy spread
- Transverse damping time
- Natural emittance

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Dissipative (Fokker-Planck) part:
Damping and diffusion due to the stochastic emission of synchrotron radiation

1. Linearize for perturbations about equilibrium

\[ F(z, p_z, \Psi, \mathcal{J}; s) = f_0(\mathcal{J})g_0(\mathcal{H}_z) + f_1(\Psi, \mathcal{J}; s)g_1(z, p_z; s) \]

Distribution function
Equilibrium
Perturbation
Overview of the Fokker-Planck analysis

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Longitudinal damping time \( \tau_z \)  
Energy spread \( \sigma_\delta \)  
Transverse damping time \( \tau_x \)  
Natural emittance

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Distribution function  
Equilibrium  
Perturbation

2. Assume that the transverse motion is described by dipole oscillations\(^\dagger\) at the (chromaticity-corrected) betatron frequency

\[ \mathcal{J} = \mathcal{D}(s) \]

Overview of the Fokker-Planck analysis

Hamiltonian part:
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3. Expand longitudinal perturbation as a sum of linear modes in longitudinal action and angle

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2. Assume that the transverse motion is described by dipole oscillations\(^\dagger\) at the (chromaticity-corrected) betatron frequency

3. Expand longitudinal perturbation as a sum of linear modes in longitudinal action and angle

4. Solve eigenvalue problem to determine normal modes and complex frequencies as a function of current and chromaticity

Linearized Fokker-Planck equation

Assumptions for distribution function imply

\[
F(z, p_z, \Psi, J; s) = f_0(J)g_0(H_z) + f_1(\Psi, J; s)g_1(z, p_z; s)
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\]

Distribution function

Equilibrium

Perturbation

\[
= \frac{e^{-\mathcal{J}/\varepsilon_0} e^{-\mathcal{I}/\langle \mathcal{I} \rangle}}{2\pi \varepsilon_0 \langle \mathcal{I} \rangle} - \mathcal{D}(s) \sqrt{\frac{1}{2} \mathcal{J} f_0'(\mathcal{J})} e^{i(\Psi + k\xi z)} e^{-i\omega_\beta s/c} e^{-i\Omega s/c} g_1(z, p_z)
\]

Transverse betatron oscillation in action-angle coordinates \((\mathcal{J}, \Psi)\)


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F(z, p_z, \Psi, J; s) = f_0(J)g_0(H_z) + f_1(\Psi, J; s)g_1(z, p_z; s)
\]

Equilibrium

\[
\frac{e^{-\mathcal{J}/\varepsilon_0} e^{-\mathcal{I}/\langle \mathcal{I} \rangle}}{2\pi \varepsilon_0} \frac{1}{2\pi \langle \mathcal{I} \rangle}
\]

Perturbation

\[
\mathcal{D}(s) \sqrt{\frac{1}{2} J f_0'(J)} e^{i(\Psi + k_{\xi}z)} e^{-i\omega_s / c} e^{-i\Omega / c} g_1(z, p_z)
\]

Transverse betatron oscillation in action-angle coordinates \((J, \Psi)\)

Head-tail phase, \(k_{\xi} = \frac{\omega_0 \xi_x}{\alpha_c c}\)


Linearized Fokker-Planck equation

Assumptions for distribution function imply

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F(z, p_z, \Psi, J; s) = f_0(J)g_0(H_z) + f_1(\Psi, J; s)g_1(z, p_z; s)
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- Distribution function
- Equilibrium
- Perturbation

\[
F = \frac{e^{-J/e_0}}{2\pi e_0} \frac{e^{-I/\langle I \rangle}}{2\pi \langle I \rangle} \mathcal{D}(s) \sqrt{\frac{1}{2} J f_0'(J)} e^{i(\Psi + k_\xi z)} e^{-i\omega_\beta s/c} e^{-i\Omega s/c} g_1(z, p_z)
\]

- Transverse betatron oscillation in action-angle coordinates \((J, \Psi)\)
- Complex mode frequency
- Longitudinal perturbation
- Head-tail phase, \(k_\xi = \frac{\omega_0 \xi_x}{\alpha_c c}\)

Linearized Fokker-Planck equation

Assumptions for distribution function imply

\[ F(z, p_z, \Psi, \mathcal{J}; s) = f_0(\mathcal{J})g_0(\mathcal{H}_z) + f_1(\Psi, \mathcal{J}; s)g_1(z, p_z; s) \]

\[ = \frac{e^{-\mathcal{J}/\varepsilon_0}}{2\pi \varepsilon_0} \frac{e^{-\mathcal{J}/\langle I \rangle}}{2\pi \langle I \rangle} D(s) \sqrt{\frac{1}{2} \mathcal{J} f_0'(\mathcal{J})} e^{i(\Psi + k_z s)} e^{-i\omega_\beta s/c} e^{-i\Omega s/c} g_1(z, p_z) \]

Transverse betatron oscillation in action-angle coordinates \((\mathcal{J}, \Psi)\)

Head-tail phase, \(k_z \equiv \frac{\omega_0 \xi_x}{\alpha_c c}\)

Linearized Fokker-Planck equation for longitudinal perturbation \(g_1\) becomes

\[ \frac{\Omega + i/\tau_x}{c} g_1(z, p_z) + i\{g_1, \mathcal{H}_z\} - \frac{2\pi I g_0(\mathcal{I})}{\gamma c I_A Z_0} \int d\hat{p}_z d\hat{z} \beta_x W_D(z - \hat{z}) e^{ik_z(\hat{z} - z)} g_1(\hat{z}, \hat{p}_z) = \frac{2i}{c\tau_z} \left[ \sigma_0^2 \frac{\partial^2 g_1}{\partial p_z^2} + p_z \frac{\partial g_1}{\partial p_z} + g_1 \right] \]


Linearized Fokker-Planck equation

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Distribution function
Equilibrium
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\[ \mathcal{D}(s) \sqrt{\frac{1}{2} \mathcal{J} f'_0(J)} e^{i(\Psi + k\xi z)} e^{-i\omega_\beta s/c} \]

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Longitudinal perturbation
Complex mode frequency
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Synchrotron motion
Contribution of dipolar wakefield
Fokker-Planck dynamics (damping and diffusion)

Imaginary part of \(\Omega\) determines stability of perturbation

Matrix theory of transverse collective instabilities

The linear problem can be solved by expanding the perturbation in terms of orthogonal modes (Sacherer’s method)

Scaled action \( I / \langle I \rangle \)

\[
g_1(\Phi, r) = \sum_{q,n} a_q^n g_q^n(r) \frac{e^{-r}}{2\pi} e^{in\Phi} = \sum_{q=0}^{\infty} \sum_{n=-q}^{\infty} a_q^n \frac{r^{n/2} L_q^n(r)}{\sqrt{(q+n)!/q!}} \frac{e^{-r}}{2\pi} e^{in\Phi}
\]

Gauss-Laguerre modes

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\]

So that the linear problem reduces to the matrix equation

Coupling matrix associated with dipolar impedance

\[
\left[ \frac{\Omega - i m \omega_s}{c} + \frac{i}{c T_x} + \frac{i(2p + m)}{c T_z} \right] a_p^m + \frac{2\pi I}{\gamma \mathcal{I}_A} \int dk \sum_{n,q} D_{p,q}^{m,n}(k + k_\xi) a_q^n = \frac{i}{2c T_z} \left( R_p^m a_{p+1}^m - T_p^m a_{p-1}^m \right)
\]


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$$\begin{bmatrix} \Omega - i m \omega_s \\ \frac{i}{c\tau_x} + \frac{i(2p + m)}{c\tau_z} \end{bmatrix} a_p^m + \frac{2\pi i}{\gamma I_A} \int dk \sum_{n,q} D_{p,q}^{m,n}(k + k_\xi) a_q^n = \frac{i}{2c\tau_z} \left( R_p^m a_{p+1}^{m-2} + T_p^m a_{p-1}^{m+2} \right)$$


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\[
\begin{bmatrix}
\Omega - m\omega_s \\
c \\
\end{bmatrix}
+ \frac{i}{cT_x} + \frac{i(2p+m)}{cT_z}
\] \( a_p^m \) + \( \frac{2\pi I}{\gamma I_A} \int dk \sum_{n,q} D_{p,q}^{m,n} (k + k_\xi) a_q^n = \frac{i}{2cT_z} \left( R_p^m a_{p+1}^{m-2} + T_p^m a_{p-1}^{m+2} \right)
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Mode-dependent synchrotron damping

Gauss-Laguerre modes

Coupling matrix associated with dipolar impedance

Diffusive mode coupling terms

Mode coefficient of (azimuthal, radial) mode number \((m,p)\), etc.


Matrix theory of transverse collective instabilities

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$$\begin{bmatrix} \frac{\Omega - m\omega_s}{c} + i \frac{(2p + m)}{cT_x} \\ \frac{2\pi I}{\gamma I_A} \int dk \sum_{n,q} D_{p,q}^{m,n}(k + k_\xi) a_q^n \end{bmatrix} + \frac{i}{2cT_z} \left( R_p^m a_{p+1}^{m-2} + T_p^m a_{p-1}^{m+2} \right)$$

This is an eigenvalue problem: truncating and numerically solving it gives normal modes that are linear superpositions of the $a_p^m$, each with a complex frequency $\Omega$.

If $\Omega$ has a positive imaginary part then the system is unstable


Physical picture of Fokker-Planck dissipation

- In the transverse plane we assumed simple dipole motion and obtained damping at the transverse damping rate
- In the longitudinal plane the effective damping depends on the mode number
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- The diffusion time $t_{\text{diff}}$ for a perturbation with characteristic scale length $\Delta p_z$ is

$$t_{\text{diff}} \sim \left( \frac{\Delta p_z}{\sigma \delta} \right)^2 \tau_z$$
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- In the transverse plane we assumed simple dipole motion and obtained damping at the transverse damping rate.
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Higher order modes are more strongly damped.
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Higher order modes are more strongly damped.

Increasing effective damping rate.

$\Delta p_z$: characteristic scale length, $\sigma$: standard deviation, $\tau_z$: characteristic time scale, $p$: mode number, $m$: mode number in the longitudinal direction.
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$$t_{\text{diff}} \sim \left( \frac{\Delta p_z}{\sigma_\delta} \right)^2 \tau_z \sim \frac{1}{2p + m} \tau_z$$

- Diffusion also results in additional coupling between modes, but this is weak.
Application to APS-U 7-bend achromat lattice with resistive wall transverse impedance model

\[ \beta_x, \beta_y, \eta_x \]

Parameters
(V_{rf} = 4.1 \text{ MV})

- \( \gamma = 6 \text{ GeV}/mc^2 \)
- \( \bar{C}_R = 1104 \text{ m} \)
- \( \alpha_c = 5.66 \times 10^{-5} \)
- \( \omega_s = 3271 \text{ Hz} \)
- \( \sigma_\delta = 0.0955 \% \)
- \( \tau_z = 14.06 \text{ ms} \)
- \( \varv_0 = 67 \text{ pm} \)
- \( \tau_x = 12.07 \text{ ms} \)

Application to APS-U 7-bend achromat lattice with resistive wall transverse impedance model

Second-order chromatic effects have been artificially set to zero in this study. (Can be included with a minor extension to the theory)

Also, we neglect the higher-harmonic rf cavity

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\[
\beta_x Z_D(k) = \eta_D \int ds \beta_x(s) \frac{\text{sgn}(k) - i}{\pi b(s)^3} \sqrt{\frac{Z_0 \rho(s)}{2 |k|}}
\]

round: \( \eta_D = 1 \)

flat: \( \eta_D = \frac{\pi^2}{24} \)

Mode coupling at zero chromaticity is very similar to that of Vlasov theory

In Vlasov picture the matrices are purely real at zero chromaticity, and two distinct real eigenvalues collide to become complex conjugates of each other.
Mode coupling is less clear for non-zero chromaticity $\xi = 0.75$
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\[ \xi = 0 \]

\[ \xi = 1.5 \]

Close to where 2-mode picture has only stable modes
Usual coupled-mode theory of two synchrotron modes describes physics at low chromaticity.
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\[ V_{rf} = 4.1 \text{ MV} \]

\[ V_{rf} = 8.2 \text{ MV} \]

BUT, two-mode theory has no unstable root if

\[ \xi_x \frac{\omega_0 \sigma_z}{\alpha_{cc}} \gtrsim 0.7 \]
Instability threshold is well predicted by the Fokker-Planck theory by including many modes.
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Vlasov theory underestimates $I_{\text{thresh}}$ by a factor of 2 at high chromaticity.
Visualization of the unstable mode at $\xi = 5$

Theory
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**Theory**
Comprised mostly of modes with $m = -4$
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elegant simulation
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Physics of collective transverse instability

Region where 2-mode theory is valid for $V_{rf} = 4.1$ MV

Region where 2-mode theory is valid for $V_{rf} = 8.2$ MV
Physics of collective transverse instability

At very low chromaticity, higher rf voltage gives larger stable current:

Larger rf $\rightarrow$ Larger synchrotron frequency

$\rightarrow$ Larger required frequency shift to merge modes

[Classic transverse mode coupling instability (TMCI)]
Physics of collective transverse instability

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For most values of chromaticity, lowering the rf increases the threshold current:
- Smaller rf → Longer bunch
- Lower peak current + larger chromatic frequency shift of $Z_{\text{transverse}}$

Unstable eigenmode is comprised of many Gaussian-Laguerre basis modes, and higher-order modes have larger Fokker-Planck damping.

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We have also compared results for the “textbook” example of a constant wake function, finding qualitatively similar behavior
Conclusions & future work

- A Fokker-Planck analysis may be required to determine stability in storage rings with significant levels of synchrotron radiation when $\xi \neq 0$
- Damping and diffusion affects finer-scale perturbations more strongly, which results in larger effective damping rates for higher-order modes
- The Fokker-Planck predictions of the instability threshold and mode shape agree well with simulation results when we know the longitudinal potential
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- We have extended this work to include quadrupolar wakefields, finding that this increases the predicted $I_{\text{thresh}}$ by 10% – 40%
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- We are in the process of extending this work to include the effects of 2nd order chromaticity, which we have found reduces the instability threshold for the APS-U
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Thank you for your attention!