COHERENT SYNCHROTRON RADIATION AND WAKE FIELDS WITH DISCONTINUOUS GALERKIN TIME DOMAIN METHODS

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Outline of Talk

▪ Motivation for the Study of Coherent Synchrotron Radiation (CSR)
▪ Maxwell’s Equations + Transformations
▪ Brief Summary of Discontinuous Galerkin (DG)
▪ Simulations of Wake Fields and CSR
  - Test Case: tapered rectangular chamber
  - Bunch Compressor Case: CSR in model of DESY BC0
▪ Summary and Future Outlook
Motivation

▪ Study the generation and propagation of CSR

▪ Some approximations:
  - Ultra-relativistic electron bunch ($\beta = 1$) along a curved planar orbit (2D orbit)
  - Consider rectangular cross-section vacuum chambers such as in a bunch compressor (2D domain)
  - Consider all boundaries as PEC (open ends possible)
  - Ignore collective effects for this work (known source terms)

▪ Goals:
  - Compute electromagnetic fields in the domain
  - Compute longitudinal wake potential
Maxwell’s Equations and Coordinates

- **Maxwell’s Equations**
  - Starting with Cartesian coordinates: \( \mathbf{R} = (Z, X, Y), \quad \tau = ct \)
  
  \[ \nabla \times \mathbf{E} = -Z_0 \frac{\partial \mathbf{H}}{\partial \tau}, \quad \nabla \times \mathbf{H} = \frac{1}{Z_0} \frac{\partial \mathbf{E}}{\partial \tau} + \mathbf{j} \]

  - Next consider a planar reference orbit (along \( Y = 0 \)):
    \[ \mathbf{R}_r(s) = (Z_r(s), X_r(s), 0) \] parameterized by arc length \( s \)
  - Define curvilinear coordinate transformation by:
    \[ \mathbf{e}_s = (Z'_r(s), X'_r(s), 0), \quad \mathbf{e}_x = (-X'_r(s), Z'_r(s), 0), \quad \mathbf{e}_y = (0, 0, 1) \]
  - Also, define signed curvature \( \kappa \) and scale factor \( \eta \) by:
    \[ \kappa(s) = Z''_r(s)X'_r(s) - Z'_r(s)X''_r(s), \quad \eta(s, x) = 1 + \kappa(s)x \]
Example of Geometry and Coordinates

- Example of mapping to curvilinear coordinates:

  - Advantage: source orbit is straight: simple modeling of source with small transverse size with DG
  - Disadvantage: only works if $\eta > 0$, problems with large $\kappa$
Source Term Definitions

- **Charge and current model for ultra-relativistic bunch**
  - In \((s, x, y)\) coordinates:
    \[
    \rho(s, x, y, \tau) = q\lambda(s - \tau)\delta(x)G(y) \\
    j(s, x, y, \tau) = qc\lambda(s - \tau)\delta(x)G(y)e_s
    \]
    with Gaussian distributions: \(\lambda(s), G(y)\)
    and Dirac distribution: \(\delta(x)\)
  - **Note:** \(\sigma_s, \sigma_y\) for \(\lambda(s), G(y)\) chosen such that source terms are supported only in the entrance region at \(\tau = 0\)
  - Other distributions can be used – can be coupled to a particle tracking code in the future
Fourier Series Decomposition

- Domain with parallel planar walls: \( y = \pm h/2 \)
  - Assuming PEC boundaries: use the Fourier series

\[
f(s, x, y, \tau) = \sum_{p=1}^{\infty} f_p(s, x, \tau) \phi(\alpha_p(y + h/2)),
\]

\[
f_p(s, x, \tau) = \frac{2}{h} \int_{-h/2}^{h/2} f(s, x, y, \tau) \phi(\alpha_p(y + h/2)) \, dy,
\]

\[
\alpha_p = \frac{\pi p}{h}, \quad \phi(\cdot) = \sin(\cdot) \text{ or } \cos(\cdot)
\]

- \( E_s, E_x, H_y, j_s, j_x \) use sine series and \( E_y, H_s, H_x, j_y \) use cosine
- If source is symmetric about \( y = 0 \) then even modes vanish
- If \( \sigma_y \ll h \), more Fourier series terms required
Initial Conditions

- With PEC boundary conditions for \( a \leq x \leq b \)

\[
\begin{align*}
E_{sp}(s, x, 0) &= 0 \\
E_{xp}(s, x, 0) &= -qZ_0cG_p\lambda(s)\Phi_p(x) \\
E_{yp}(s, x, 0) &= -qZ_0cG_p\lambda(s)\Psi_p(x) \\
H_{sp}(s, x, 0) &= 0 \\
H_{xp}(s, x, 0) &= qcG_p\lambda(s)\Psi_p(x) \\
H_{yp}(s, x, 0) &= -qcG_p\lambda(s)\Phi_p(x)
\end{align*}
\]

\[
\begin{align*}
\Phi_p(x) &= \sinh(\alpha_p b) \frac{\cosh(\alpha_p(x-a))}{\sinh(\alpha_p(b-a))} - \cosh(\alpha_p x)\Theta(x) \\
\Psi_p(x) &= \sinh(\alpha_p b) \frac{\sinh(\alpha_p(x-a))}{\sinh(\alpha_p(b-a))} - \sinh(\alpha_p x)\Theta(x)
\end{align*}
\]
Combining All Transformations

- **Issue**: how to evaluate $\delta(x)$ in $\partial E_{sp}/\partial \tau$ equation?
- **Fix**: replace $H_{yp}$ by $\tilde{H}_{yp} = H_{yp} - q c G_p \lambda(s - \tau) \Theta(x)$
- **Result:**
  - ✓ Maxwell’s Eqs.
  - ✓ C. Transform
  - ✓ Source Def.
  - ✓ F. Decomp.
  - ✓ Smoother Src.
  - ✓ Initial Conds.
Brief Summary of DG

- Discontinuous Galerkin Finite Element Method
  - General idea: approximate each field by an $N_{th}$ order polynomial in $(s, x)$ on a triangular element
  - Fields are not imposed to be continuous across edges
  - Derivatives are computed for each element independently
  - Elements are coupled with a flux function
  - Naturally handles discontinuities if they exist along element edges such as $\Theta(x)$ along $x = 0$
  - Great for hyperbolic problems and easily parallelized
  - Well-suited for GPU computing since operators are dense
Final Steps for the Numerical Method

- **Additional Notes:**
  - Evolve fields with 4th order low-storage RK
  - Important: align elements along $x = 0$ and where $\kappa$ is discontinuous (i.e. when using piecewise-defined orbits)
  - Sum over $p$ modes for full 3D solution
  - Solution can easily be remapped to Cartesian coordinates
  - Define longitudinal wake function by:

\[
w_s(z) = -\frac{1}{q} \int_0^T E_s(\tau - z, 0, 0, \tau) d\tau
\]

\[
= -\frac{1}{q} \sum_{p=1}^{p_{\text{max}}} \sin \left( \frac{\pi p}{2} \right) \int_0^T E_{sp}(\tau - z, 0, \tau) d\tau
\]
Test Case Simulation

▪ Test Case
  - Straight wave-guide with a taper
  - Geometry generates wake
  - $E_{sp}$ sampled along $x = 0$ near source, sum over $p = 1, \ldots, 9$
  - Comparison to solution from CST Particle Studio™
Test Case Wake Function

$\sigma_s=10$ mm, $h=50$ mm
$\sigma_y=1$ mm, $T=6$ m

- DG: $N=8$, $K=7190$, $p_{\text{max}}=9$
- CST: $\sim 100$M mesh cells
Test Case Wake Simulation
Bunch Compressor Simulation

- Bunch Compressor Case
  - DESY BC0 – assume piecewise constant curvature
  - CSR and geometry generates wake
  - $E_{sp}$ sampled along $x = 0$ near source
Bunch Compressor Wake Function

$\sigma_s = 2\, \text{mm}, \ h = 20.3\, \text{mm}$
$\sigma_y = 1\, \text{mm}, \ T = 6\, \text{m}$

DG: $N = 6, \ K = 114872, \ p = 1$
Bunch Compressor Wake Simulation
Bunch Compressor Field Simulation

\[ E_s | \tau = 0.00 \text{ m} \]
Summary and Future Outlook

- Developed 2D time-domain field solver for ultra-relativistic bunches with a curvilinear transformation and Fourier decomposition using DG
- GPU-enabled MATLAB code built on DG Methods from “Nodal Discontinuous Galerkin Methods” by J. Hesthaven, T. Warburton
- Future Outlook:
  - Couple computed wake fields with particle tracking
  - Estimate wall losses with Poynting flux on walls
  - Additional validation with other EM field or CSR codes
Thank you for your attention!

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