1. INTRODUCTION

The simulation code for single-bunch instabilities, SISR(Single-Bunch Instabilities in Storage Rings) was developed and applied to study instabilities in the SPring-8 storage ring. SISR uses complex amplitude of betatron motion and it enable to treat both distributed broad-band impedance such as small discontinuities of a beam pipe and localized impedance like cavities.

2. EQUATIONS OF MOTION

The transverse motion of i-th electron under the force \( F \) is represented with \( \eta_i \) as

\[
\eta_i = \frac{x_i}{\beta_i}, \quad \theta = \frac{1}{\nu_0} \int_0^s \frac{d\xi}{\beta_i}
\]

\[
\frac{d^2 \eta_i}{d\theta^2} + \left( \nu_0 + \Delta \nu_i \right)^2 \eta_i = \nu_0 \beta_i^2 F_i
\]

where \( \nu_0, \beta \) and \( E \) is the betatron tune, the beta function and the energy, respectively.

The force \( F_i \), from wake field is

\[
F_i = \epsilon \sum_{j=1}^{N_p} q_j a_j \frac{d}{ds} W^{-1}(z_j - z_i, s)
\]

\[
= \epsilon \beta_i^2 \sum_{j=1}^{N_p} q_j \eta_j \frac{d}{ds} W^{-1}(z_j - z_i, s)
\]

Assuming that

\[
\left| \frac{d^2 \alpha_i}{d\theta^2} \right| << 2 \nu_0 \frac{d\alpha_i}{d\theta}
\]

(4), which means that the typical varying time of \( \alpha \) is much smaller than the betatron period \( \lambda \beta/c \), and using the phasers shown below,

\[
\eta_i = \text{Re} \left\{ a_i(\theta) e^{\nu_0 \theta} \right\} = \frac{1}{2} \left\{ a_i(\theta) e^{\nu_0 \theta} + c.c. \right\}
\]

\[
F_i = \text{Re} \left\{ f(\theta) e^{\nu_0 \theta} \right\} = \frac{1}{2} \left\{ f(\theta) e^{\nu_0 \theta} + c.c. \right\}
\]

\[
f(\theta) = \epsilon \beta_i^2 \sum_{j=1}^{N_p} q_j a_j \frac{d}{ds} W^{-1}(z_j - z_i, s)
\]

the equation (2) become

\[
\frac{d \alpha_i}{d\theta} = i \Delta \nu_i \alpha_i + \frac{\nu_0}{2\epsilon E_0} \int_0^{s_0} \beta_i^2 F_i \frac{d\theta'}{\nu_0}
\]

\[
+ \frac{\nu_0}{2\epsilon E_0} \int_0^{s_0} \beta_i^2 \frac{d\theta'}{\nu_0} e^{-2 i \nu_0 \theta'} \frac{d\theta'}{\nu_0}
\]

(8).

The third term of the right hand side can be set to zero in usual cases because \( \beta_i^2 F \) usually does not have the Fourier component of the frequency \( 2\nu_0 \).

Hence the equation

\[
\frac{d \alpha_i}{d\theta} = i \Delta \nu_i \alpha_i + \frac{\nu_0}{2\epsilon E_0} \int_0^{s_0} \beta_i^2 F_i \frac{d\theta'}{\nu_0}
\]

(9) is used to simulate distributed impedance.

The integral in this equation (9) can be written as

\[
\int_0^{s_0} \beta_i^2 F_i \frac{d\theta'}{\nu_0} = \frac{1}{\lambda} \beta_i \int_0^{s_0} \frac{d\theta'}{\nu_0} \int \sum_{j=1}^{N_p} \beta_j W_k(z_j - z_i)
\]

and from the approximation (4), we have

\[
\int_0^{s_0} \beta_i^2 F_i \frac{d\theta'}{\nu_0} = \sum_{k=1}^{N_p} \beta_k \int_0^{s_0} f_k d\theta'
\]

(10).

3. DIFFERENCE EQUATIONS

The difference equations used in CISR to simulate the electron motion are as follows. For longitudinal motion, \( \Delta E_i = E_i - E_0 \) and \( z = c t \), where \( c \) is the co-ordinate in the direction of particle, is used to describe longitudinal motion of particles. \( T_n \) and \( E_0 \) are the revolution period and reference energy, respectively.

In the following, the symbol with superscript + and - are the value after and before the passage of each element. In the following, respectively.

3.1 Lattice with Distributed Broad-Band Impedance

\[
\Delta E_i^+ = \Delta E_i^- = \Delta E_i \frac{T_n}{E_0} \left( 1 + \frac{\Delta E_i^+}{E_0} \right)^2
\]

(11)

\[
z_i^+ = z_i^- + \alpha \Delta E_i^+ \frac{E_0}{E_i} \Delta T
\]

(12)

\[
r_i^+ = r_i^- + \text{Re} \left\{ f^+ e^{i \phi_{i,m}} \right\} \Delta \theta
\]

(13)

\[
\phi_i^+ = \phi_i^- + \Delta \nu_i \Delta \theta + \frac{\text{Im} \left\{ g_i^+ e^{i \phi_i^-} \right\}}{|r_i^- + r_i^+|^2} \Delta \theta
\]

(14)

\[
g_i^\pm = \frac{1}{\Delta T} \sum_{j=1}^{N_p} q_j a_j \sum_{k=1}^{N_l} \beta_k W_k(z_j^\pm - z_i^\pm)
\]

(15)
where $\Delta \theta = 2 \pi \frac{\Delta T}{\tau_0}$, $r = |a|$, $a = re^{i \theta}$.

### 3.2 Localized Broad-Band Impedance

For the localized impedance such as cavities,

$$\Delta E_i = \Delta E_{i0} + \sum_{j=1}^{N_i} q_j W_i(z_i - z_{i0})$$

$$a_i^j = a_i^0 + \sum_{k=1}^{N_k} \beta_k^j z_{j0} \sin^2 \left( \frac{2 \pi f_j}{c} t + \phi_j \right)$$

where $x = \sqrt{\beta} \eta_i = \sqrt{\beta} \Re \left[ a_i(\theta) e^{i \phi \theta} \right]$

### 3.3 Acceleration

For acceleration by $eV_i(z) = eV_i \sin \left( 2 \pi f_j \frac{z}{c} + \phi_j \right)$,

$$\Delta E_i^* = \Delta E_{i0}^* + eV_i \sin \left( 2 \pi f_j \frac{z}{c} + \phi_j \right)$$

$$a_i^* = a_i^0 \pm i \frac{e V_i}{E_0} \Im \left[ a_i^0 e^{i \phi_{i0}} \right] e^{i \phi_{i0}}$$

Eq.(25) is transverse radiation damping.

### 3.4 Radiation Excitation

$$\Delta E_i^* = \Delta E_{i0}^* + \sqrt{\left( \frac{4 \Delta T_0}{\tau_0} \right) \left( \sigma_{t0}^0 \right)} u_i$$

$$a_i^* = a_i^0 \pm \sqrt{\left( \frac{4 \Delta T_0}{\tau_0} \right) \epsilon_i} e^{i 2 \pi w t_i}$$

where $u_i, v_i$ are the Gaussian random number and $w_i$ is uniform random number. $\Delta T_0$ is the time difference between each excitation and $\tau_0$ and $\sigma_{t0}$ is radiation damping time for $E_0$ and $x_i$, respectively. $\sigma_{t0}$ and $\epsilon_i$ are natural energy spread and emittance.

### 4. PARTICLE-IN-CELL METHOD

SISR is the particle code and a Particle-In-Cell(PIC) method is used to make wake field and interact the particle with the wake field. The shape function used in SISR is

$$S(x) = \left\{ \begin{array}{ll}
\frac{1}{2} \left[ 1 - \left( \frac{3}{\Delta x} \right)^2 \right] & \text{if } |x| \leq \frac{3}{2} \Delta x \\
\frac{1}{2} \left[ 1 - \left( \frac{\Delta x}{3} \right)^2 \right] & \text{if } \frac{3}{2} \Delta x < |x| \\
0 & \text{otherwise}
\end{array} \right. \quad (22)$$

and the distributions

$$\begin{align*}
\rho(z_p) &= \sum_{i=1}^{N} a_i q_i S(z_i - z_p) \\
p(z_p) &= \sum_{i=1}^{N} a_i q_i S(z_i - z_p)
\end{align*} \quad (23)$$

are evaluated on the mesh points whose position is represented by $z_p$.

And wake potentials appeared in above equations are evaluated at the mesh points using these distributions as

$$E_{\text{wake}}(z_p) = \sum_{j=1}^{N_q} q_j \sum_{k=1}^{N} \beta_k^j W_k^j(z_j - z_p)$$

$$E_{\text{wake}}(z_p) = \pm \int_{z'} a_i^p(z') \sum_{k=1}^{N_k} \beta_k^j W_k^j(z' - z) \, dz'$$

### 5. WAKE FUNCTIONS

The three types of impedance models listed below are used to get wake functions. For longitudinal,

$$Z_L^0 = z_0 + i \frac{Z_L^0}{z_0} \frac{Z_L^0}{z_0} \frac{Z_L^0}{z_0} (1 + i)$$

$$W_L = Z_L^0 \frac{\delta(z)}{\delta z} Z_L^0 \frac{\delta(z)}{\delta z} Z_L^0 \frac{\delta(z)}{\delta z} (1 + i)$$

and for transverse,

$$Z_T^0 = z_0 + i \frac{Z_T^0}{z_0} \frac{Z_T^0}{z_0} \frac{Z_T^0}{z_0} (1 + i)$$

$$W_T = Z_T^0 \frac{\delta(z)}{\delta z} Z_T^0 \frac{\delta(z)}{\delta z} Z_T^0 \frac{\delta(z)}{\delta z} (1 + i)$$

### 6. THE SPRING-8 STORAGE RING

The CISR is applied to the study of the instabilities of the SPRING-8 storage ring. The parameters of the ring is shown in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>$E_0$</td>
<td>8 GeV</td>
</tr>
<tr>
<td>Revolution Frequency</td>
<td>$T_0$</td>
<td>208.77 kHz</td>
</tr>
<tr>
<td>Energy Loss per Turn</td>
<td>$U_0$</td>
<td>9.2 MV</td>
</tr>
<tr>
<td>Damping Partition Numbers</td>
<td>$J_0 / J_1$</td>
<td>2 / 1</td>
</tr>
<tr>
<td>Momentum Compaction Factor</td>
<td>$\alpha$</td>
<td>$1.41 \times 10^{-4}$</td>
</tr>
<tr>
<td>Betatron Tune (vertical)</td>
<td>$\nu_v$</td>
<td>16.16</td>
</tr>
<tr>
<td>Averaged Betatron Function</td>
<td>$\beta$</td>
<td>17.3 m</td>
</tr>
</tbody>
</table>

The impedance of the ring is estimated in ref.[1] and is

$$Z^0 = -1.67 \times 10^5 \, \text{ohm} + 400 + 1.49 \times 10^9 \left( 1 + \frac{i}{\sqrt{90}} \right)$$

$$Z^0 = -2.13 \times 10^3 \, \text{ohm} + 4.98 \times 10^14 + 4.21 \times 10^{10} \left( 1 + \frac{i}{\sqrt{90}} \right)$$

where $Z^0$ is vertical transverse impedance. The unit for them are $\Omega$ and $\Omega / m$, respectively.

### 6.1 Longitudinal Instabilities

Figure 1 shows the dependence of the bunch length and the energy spread $\sigma_i / E$ on the bunch current $I_b$. This ring is rather inductive compared with colliders, the potential-well distortion lengthen the bunch length and the threshold of microwave instabilities can not be seen until the threshold current of the transverse instabilities mentioned later.
6.2 Transverse Instabilities

Figure 2 shows the bunch current increase vs. time used in the simulation.

Figure 2. bunch current shape vs. time

Figure 3 and Figure 4 show the amplitude of betatron motion of the bunch vs. time for chromaticity $\xi=0$ and $\xi=4$, respectively. Instabilities occurs at $I_b=3\,\text{mA}$ and $I_b=7\,\text{mA}$ for $\xi=0$ and $I_b=10\,\text{mA}$ for $\xi=4$.

Figure 3. Amplitude of the betatron motion vs. time. $\xi=0$.

Figure 4. amplitude of the betatron motion vs. time. $\xi=4$.

Figure 5,6 are the spectrum of the betatron motion of the bunch. (m,n)=(0,0) and (m,n)=(0,1) mode can be seen. The m=1 mode must be exist at the synchrotron frequency $f_s=1.5\,\text{kHz}$ lower position, around $f=32\,\text{kHz}$, but no signal can be seen.

From Figure 5, which is for $\xi=0$, the coupling of mode (m,n)=(0,0) and (m,n)=(1,0) occurs at $I_b=3\,\text{mA}$ and the coupling of mode (m,n)=(0,0) and (m,n)=(2,0) occurs at $I_b=7\,\text{mA}$. Both lead to instabilities. For chromaticity $\xi=4$, of which data is not shown here, the signal of mode (m,n)=(0,0) can not been.

In Figure 4, No instabilities occurs near $I_b=3\,\text{mA}$, but the m=2 mode growths up at $I_b=10\,\text{mA}$. The difference between $\xi=0$ and $\xi=4$ seems to be from the effect of the head-tail damping, which can be seen at the beginning of the bunch current increase, where time ~5ms in Figure 4 and it is faster than radiation damping time which is seen for $I_b=0\,\text{mA}$.

7. CONCLUSION

The simulation code for single-bunch instabilities was developed and applied to the SPring-8 storage ring. No longitudinal microwave instabilities can not seen and the threshold of the transverse instabilities is a few mA and the positive chromaticity increase the threshold current.

8. REFERENCES