A Geometrical Horizontal-Vertical Coupling

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Abstract
In a circular accelerator, a horizontal-vertical coupling can occur just by a geometrical reason without any explicit coupling elements, skew quadrupole magnets etc. The coupling is represented by a rotation (SO(2)) between horizontal and vertical coordinates (and momenta). The angle is unique and intrinsic to a given configuration of the reference orbit. This effect, related to the non-integrability of the parallel transport, we give a formula for it. Sect.4 is devoted to the brief study of the twist angle. This angle is related to the tune and emittance changes.

1 INTRODUCTION
The betatron coordinates are defined with respect to the orbit. The reference orbit closes after one turn: the horizontal and vertical coordinates (and momenta). The angle corresponding to bending magnets and straight lines.

1.1 GLOBAL COORDINATE SYSTEM
Let the reference orbit be described by \( r(s) \), where \( s \) is the arc length: \( |dr| = |ds| \). We introduce the unit tangent vector \( e_z(s) = \hat{r}(s) \). Letting \( \Omega = e_z \times e_z \), one has:

\[
\dot{e}_z = \Omega \times e_z.
\] (1)

Now, at a certain point in the ring, say \( s = 0 \), we define two vectors \( e_x(0) \) and \( e_y(0) \) perpendicular to \( e_z(0) \) in a way that \( (e_x(0), e_y(0), e_z(0)) \) forms a right-handed orthonormal basis. We can define \( e_x(s) \) and \( e_y(s) \) by the parallel transport equation[3]:

\[
\dot{e}_i = \Omega \times e_i, \quad (i = x, y, z),
\] (2)

with the initial condition at \( s = 0 \). We can define the curvature radii \( \rho_x \) and \( \rho_y \) as \( \Omega = -e_y/\rho_x + e_x/\rho_y \), so that Eq. (2) can be put in a form:

\[
\begin{pmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & \rho_x^{-1} \\
0 & 0 & \rho_y^{-1} \\
-\rho_x^{-1} & -\rho_y^{-1} & 0
\end{pmatrix}
\begin{pmatrix}
e_x \\
e_y \\
e_z
\end{pmatrix}.
\] (3)

Its integration defines the coordinate system:

\[
(e_x(s), e_y(s), e_z(s))^t = F(s, 0)(e_x(0), e_y(0), e_z(0))^t
\]

Here \( F(s, 0) = T \int_0^s \exp A(s, \rho_x(s), \rho_y(s)) ds \), where \( T \) stands for the time ordered product and \( A \) is the matrix in Eq.(3). With \( (e_x, e_y) \) thus defined, we can define the coordinate system for all \( s \in [0, C] \), where \( C \) is the ring circumference. Since the orbit should be smooth, we have \( e_z(C) = e_z(0) \), implying

\[
F(C, 0) =
\begin{pmatrix}
\cos \Phi & \sin \Phi & 0 \\
-\sin \Phi & \cos \Phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (4)

We define the twist angle \( \Phi \) by this equation. Now, \( F(C, 0) \) can be expressed as a product of transformations for magnets: \( f_n f_{n-1} \cdots f_1 \), where \( f_i \) stands for the transport through the \( i \)-th magnet:

\[
f_i = \exp \left\{ A(\rho_x^i, \rho_y^i) l_i \right\}.
\] (5)

Here \( l_i \) is the length of the reference orbit in it.

In Fig.1, we have shown an example of the parallel transport and the twist.

2 THE INTRODUCTION AS A NON-INTEGRABILITY
The existence of the twist angle \( \Phi \) is related to the geometrical nature of the reference orbit. The function \( \hat{e}_z(s) \) defines a map from the reference orbit, \( [0, C] \), to a unit surface \( S^2 \), making a closed curve \( \Gamma \) on it. From Eq.(1), \( \Gamma \) is (piecewisely) geodesic on \( S^2 \).

At each point on \( \Gamma \), we can attach \( e_z(s) \) (on its tangential plane). The twist \( \Phi \) is then the angle between \( e_x(0) \) and
bending and quadrupole magnets are chosen in such a way that linear betatron oscillations, coherent dynamics etc. Here, we study a similar case with linear betatron oscillation. Assume that the magnet number 6 represents a long planar arc. The e\(_6\) field is attached to the each point on \(\Gamma\). The curve corresponds to the configuration in Fig.1: At \(s=0\), the \(e_6\) is at P in this figure and \(e_6\) (the dotted arrow) is parallel to the “equator”. At the magnet -1, \(e_6\) moves to Q with \(e_6\) parallally transported. In the same way, it passes the magnets-2 and -3. Then \(e_6\) comes back to P. It makes a round trip either after the magnets-4 and -5, from P to S (no contribution to \(\Phi\)), or after the magnet-6, around the equator (add 2\(\pi\) to \(\Phi\)). In the end, at P, \(e_6\) is tilted by \(\Phi=\pi/2\). The solid angle is \(\pi/2 +2\pi\) (\(\equiv \Phi, mod\ 2\pi\)).

where \(R\) is the rotation in Eq.(7), \(m\)’s are \(2 \times 2\) symplectic matrices. As an example, let us assume

\[
m_{x,y} = \begin{pmatrix} \cos 2\pi \nu_{x,y} & \beta_{x,y} \sin 2\pi \nu_{x,y} \\ -\beta_{x,y}^{-1} \sin 2\pi \nu_{x,y} & \cos 2\pi \nu_{x,y} \end{pmatrix},
\]

as a generalization of the Möbius ring[5], which corresponds to \(\Phi=\pi/2\). By the standard technique[6], one can calculate the eigenvalues of \(M(\Phi)\). The unstable region in the \((\nu_x, \nu_y)\) plane is shown in Fig. 3 for \(\Phi=0.02 \times 2\pi\) and \((\beta_x, \beta_y) = (0.33, 0.01)m\). It is rather surprising that a small \(\Phi\) gives rise to a large unstable region.

Let us discuss the equilibrium beam envelope[7] for an electron ring. Following the treatment of radiation made in Ref.[8], the equilibrium envelope matrix \(\Sigma_{ij} = <\xi_i \xi_j>\),

\[
\Phi = 2\pi - \sum_i \alpha_i = the solid angle subtended by \(\Gamma\). (6)
\]

This is Gauss-Bonnet theorem[4]. The solid angle is defined with its orientation. Note that \(\Phi\) is defined in \(mod\ 2\pi\). This angle reflects the non-integrability of \(e_6\) on \(\Gamma\) and is an example of the holonomy group on \(S^2\). Note that \(\Phi\) does not depend on the way of defining the coordinate frame. It is a physical object. It is easily seen that \(\Phi\) does not change when we start from a point different from \((s=0)\). An “additional” rotation should be applied somewhere in the ring so that the one turn matrix \(F(C,0)\) becomes the identity matrix.

4 DYNAMICAL EFFECTS

When \(\Phi\) exists, the \((x, y)\) coordinates should be adjusted at \(s=0\):

\[
\begin{pmatrix} x \\ y \end{pmatrix}_{0+} = \begin{pmatrix} \cos \Phi & \sin \Phi \\ -\sin \Phi & \cos \Phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_{0-}. \tag{7}
\]

This should be applied no matter what dynamics is concerned, linear and nonlinear betatron oscillations, polarizations, coherent dynamics etc. Here, we study a simple case with linear betatron oscillation. Assume that the bending and quadrupole magnets are chosen in such a way that the \(\rho\)’s are either horizontal or vertical with respect to \((e_6(s), e_6(s))\). Then the transfer matrix is always in a piecewise diagonal form. The one turn matrix should be in the form

\[
M(\Phi) = R(\Phi) diag(m_x, m_y), \tag{8}
\]

\[
\begin{pmatrix} \cos \pi \nu_{x,y} & \beta_{x,y} \sin \pi \nu_{x,y} \\ -\beta_{x,y}^{-1} \sin \pi \nu_{x,y} & \cos \pi \nu_{x,y} \end{pmatrix},
\]

as a generalization of the Möbius ring[5], which corresponds to \(\Phi=\pi/2\). By the standard technique[6], one can calculate the eigenvalues of \(M(\Phi)\). The unstable region in the \((\nu_x, \nu_y)\) plane is shown in Fig. 3 for \(\Phi=0.02 \times 2\pi\) and \((\beta_x, \beta_y) = (0.33, 0.01)m\). It is rather surprising that a small \(\Phi\) gives rise to a large unstable region.

Let us discuss the equilibrium beam envelope[7] for an electron ring. Following the treatment of radiation made in Ref.[8], the equilibrium envelope matrix \(\Sigma_{ij} = <\xi_i \xi_j>\),
with $\xi = x, x', y, y'$, satisfies the equation:

$$\Sigma = \lambda M \Sigma M^t + (1 - \lambda)\epsilon_0 RBR^t,$$

$$B = \text{diag}(\beta_x, 1/\beta_x, 0, 0).$$

Here $\lambda$ is a constant $\leq 1$ and $\epsilon_0$ is the nominal equilibrium emittance. We have assumed that the “vertical diffusion” is ignorable compared to the “horizontal” one. At the equilibrium, the emittance is computed as:

$$\epsilon_{x,y} = \text{Abs}[\text{Eigenvalues}[J\Sigma]],$$

where $J$ is the $4 \times 4$ symplectic metric. In Fig. 4 we plot the emittances and the growthrate-1 (top) and the beam sizes (bottom) vs. $\Phi$ for the same parameters of Fig. 3. For $\Phi \in [0,8,1,1] \cup [1,7,2,3]$, the instability occurs.

Note that the instability pattern and the emittances are identical when we make transformations $(\nu_x, \nu_y) \rightarrow (\nu_x + 1/2, \nu_y + 1/2)$ or $\Phi \rightarrow \Phi + \pi$.

Figure 4: (Top) Emittances and growthrate-1 (dashed line), (bottom) $\Sigma_{11}$ (solid line), $\Sigma_{13}$ (dashed line), $\Sigma_{33}$ (dotted line), as functions of $\Phi$ for $(\nu_x, \nu_y) = (0.2, 0.15)$ and $(\beta_x, \beta_y) = (0.33, 0.01)$m. All are normalized by $\epsilon_0$.

5 DISCUSSIONS

The main point of this paper was to point out the presence of the intrinsic $SO(2)$ rotation associated with the configuration of the reference orbit. The twist angle $\Phi$ is unique for a given reference orbit and is a physical object which one should include in tracking etc.

Here, it seems appropriate to discuss the “geometry”. Is the origin of $\Phi$ geometrical or dynamical? When the bending angle $\theta$’s are small, we can also use a planar orbit as the reference. In this case, $\Phi$ does not appear as the geometrical effect of the reference orbit. The same x-y coupling, however, will appear physically, nevertheless. This can be thought of dynamical coupling because it comes from the “Hamiltonian” but it can still be called geometrical because it comes from the Minkowski nature of the space-time.

In this paper, we have discussed the simple dynamical effects only. Further study is needed to clarify the effects in nonlinear dynamics, polarization, coherent instabilities and so on.

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6 REFERENCES


A DISCONTINUITY OF FRENET-SERRET TRIAD

The parallel transport is not the only way to define the coordinates on the reference orbit. Here, we show how $\Phi$ emerges when we use the Frenet-Serret triad (FST). FST is defined locally. On a straight line, FST is not defined so that we define it by the parallel transport from the last curve. If the oscillating planes ($x$-direction, represented by $e_{FS}^x$) of the next and previous curves are not parallel, it should exhibit abrupt rotation[2].

Figure 5: The $x$ coordinates (arrows) according to the Frenet-Serret triad for the same example of Fig.1.

Let us define the angle to adjust $e_{FS}^x$’s between magnets no.2 and i+1 as $\delta_i$:

$$(e_{FS}^x, e_{FS}^y)_{(i+1)-} = R(\delta_i) (e_{FS}^x, e_{FS}^y)_{i+}.$$