TRANSFER MATRIX OF LINEAR FOCUSING SYSTEM IN THE PRESENCE OF SELF-FIELD OF INTENSE CHARGED PARTICLE BEAM

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Abstract
The computation algorithm of the transfer matrix in the presence of self-field of the intense charge particle beam is given.

INTRODUCTION
Within the framework of moment method [1] the computation algorithm of the transfer matrix in the presence of self-field of the intense charge particle beam is given. The transfer matrix depends on both the linear external electromagnetic field parameters and the initial value of the second order moments of the beam distribution function. In the case of coupled degrees of freedom the independent 2D subspaces of the whole phase space are found by means of the linear transformation of the phase space variables. The matrix of this transformation connects with second order moments of the beam distribution function. The momentum spread of the beam is taken into account also.

BASIC EQUATIONS
Let us consider the vector $Y^T = \begin{pmatrix} x_1, x_2, x'_1, x'_2 \end{pmatrix} = \begin{pmatrix} X^T & V^T \end{pmatrix}$, where superscript $T$ defines transpose vector or matrix, prime denotes derivative with respect to distance $s$ along the beam trajectory. In the linear approximation the vector $Y$ satisfies to matrix equation:

$$Y' = AY$$

Here $E_s$ (n=2) is unit matrix of n-th order, $a$ and $b$ are 2x2 matrices defined by both external electromagnetic fields $a_{ext}$, $b_{ext}$ and beam self-field $b_s$:

$$a = a_{ext}, \quad b = b_{ext} + b_s$$

In the presence of the longitudinal electric field $E_s$ system (1) must be added by equation for longitudinal momentum $p$:

$$p' = \frac{Ze}{Ac} E_s - \frac{1}{Bp} E_s\phi_p, \quad \phi_p = v_p/c - \text{relativistic velocity of the beam},$$

where $\beta_p = v_p/c$ - speed of light, $e$ - unit charge, $Z,A$ - ion charge and mass.

Matrix $b_s$ depends on the beam RMS-dimensions [1].

Let us define the second order moments $M$ of the beam distribution function $f$:

$$M = YY^T = \frac{1}{N} \int Y Y^T f dV \quad \text{(4)}$$

Here $N$ is number of particle, integration in (4) is fulfilled over all phase space occupied by particles. In accordance with system (1) matrix $M$ satisfy the equation [1]:

$$M' = AM + MA^T \quad \text{(5)}$$

ROTATING FRAME
For simplification of system (1) it is possible to eliminate matrix $a$. Let us introduce new phase space variables $Y'_R$ by means of linear transformation:

$$Y = R_0 Y_R \quad R_0 = \begin{pmatrix} Q & 0 \\ Q' & Q \end{pmatrix}$$

with 2x2 matrix $Q$. By substituting (6) into (1) we have:

$$Y'_R = A_R Y_R \quad A_R = R_0^{-1} A R_0 - R_0^{-1} R_0'$$

By representing matrix $A_R$ (7) in block form one can get:

$$A_R = \begin{pmatrix} E_2 & b_R \\ 0 & a_R \end{pmatrix}$$

$$a_R = Q^{-1} (-2Q' + aQ) \quad b_R = Q^{-1} (bQ + aQ' - Q')$$

In the case $a_R = 0$ we have:

$$Q' = \frac{1}{2} aQ \quad b_R = Q^{-1} \left( b + \frac{1}{4} a^2 - \frac{1}{2} a' \right) Q$$

For general focusing system with longitudinal electromagnetic fields $E_s$, $B_s$, dipole magnets and quadrupole lenses the matrices $Q$ and $b_R$ have the following form ($p_0$ is the initial value of momentum $p$):

$$Q = \sqrt{\frac{p_0}{p}} Q_0 \quad Q_0 = \begin{pmatrix} \cos \varphi_B & \sin \varphi_B \\ -\sin \varphi_B & \cos \varphi_B \end{pmatrix}, \quad \text{(10.1)}$$

$$\varphi_B' = k = \frac{1}{2Bp}$$

$$b_{R_{ext}} = Q'_{ext} Q_0 - k^2 E_s - k^2 \left( 3 - 2 \beta_p^2 \right) \left( \frac{E_s}{B_p B_s} \right)^2 E_2$$

Here matrix $b_{ext}$ is defined by gradients of the quadrupole lenses $G(s)$ and a bending radius of the dipole magnets $\rho_M(s)$:

$$b_{ext} = \begin{pmatrix} G(s) + 1 & 0 \\ -\rho_M(s) & \frac{G(s)}{Bp} \end{pmatrix}$$

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Matrix of the second order moments $M$ (4) is connected with one defined in the rotation frame $M_R$ by the following manner:

$$M = R_0 M_R R_0^T$$  \hspace{1cm} (11)

**BEAM SELF-FIELD**

Influence of the beam self-field leads to dependence of the matrix $b_{rs}$ on RMS dimensions [1]:

$$b_{rs} = \frac{p_0 Z I}{p A I_A (\beta_p \gamma_p)^3} M_{xx}^{1/2}$$  \hspace{1cm} (12)

Where $I$ - beam current, $I_A$ - Alfven's current, $\gamma_p$ - relativistic factor. Matrix $M_R^{1/2} M_R^{1/2}$ is defined as:

$$M_{xx}^{1/2} M_{xx}^{1/2} = M_{xx} = X_R X_R^T$$  \hspace{1cm} (13)

**TRANSFER MATRIX**

Assuming all calculations are made in the rotational frame the notation “$R$” will be dropped in the successive expressions. Let us introduce matrix $\Lambda$ in accordance with equation (7):

$$\Lambda' = A \Lambda \quad A = \begin{pmatrix} 0 & E_2 \\ b & 0 \end{pmatrix}$$  \hspace{1cm} (14)

The product $\Lambda \Lambda^T$ satisfies to equation (5) for matrix $M$: For this reason equality $M = \Lambda \Lambda^T$ will be valid at arbitrary point $s$ if the same condition is valid at initial point of the system.

Transfer matrix $R$ of system (14) may be found as:

$$R = \Lambda \Lambda_0^{-1}$$  \hspace{1cm} (15)

The solution $Y$ of the equations (7) and matrix $M$ are defined by matrix $R$ in the standard form:

$$Y = R Y_0 \quad M = R M_0 R^T,$$  \hspace{1cm} (16)

where index “0” denotes initial values of the variables.

In computer calculations it is convenient to represent matrices $R$ and $M$ in the block form:

$$R = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \quad M = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix}$$  \hspace{1cm} (17)

In accordance with (14) $2 \times 2$ matrices $C$ and $S$ satisfy to the system of second order differential equation:

$$C' = b C \quad \begin{pmatrix} C_0 \\ C_0' \end{pmatrix} = \begin{pmatrix} E_2 \\ 0 \end{pmatrix}$$  \hspace{1cm} (18)

$$S' = b S \quad \begin{pmatrix} S_0 \\ S_0' \end{pmatrix} = \begin{pmatrix} E_2 \\ 0 \end{pmatrix}$$

The equations for matrices $C$ and $S$ are not independent. They are connected by expression for matrix $M_{xx}$ defined the matrix $b_{ss}$ (12) for beam self-field:

$$M_{xx} = C M_{xx0} C^T + S M_{xx0} S^T + C M_{xx0} S^T + S M_{xx0} S^T$$  \hspace{1cm} (19)

Thus the elements of transfer matrix $R$ satisfy to the nonlinear differential equations and its solutions depend on initial value of the second order moments.

**INDEPENDENT SUBSPACES**

Let us introduce new phase space variables $Y_i$:

$$Y = T Y_i \quad T = \begin{pmatrix} t_x & 0 \\ t_{xy} & t_v \end{pmatrix}$$  \hspace{1cm} (20)

In accordance with formulae (7) and (20) vector $Y_i$ satisfies to the equation (7) with matrices $A_1$ depending on elements of matrix $T$ and its derivative. By postulating the antisymmetry of matrix $A_1$ one can get the equations for elements of matrix $T$:

$$A_1 = \begin{pmatrix} a_x & t_x^{-1} t_v \\ -1 t_x^{-1} t_v^T & a_v \end{pmatrix}$$  \hspace{1cm} (21)

$$t_x + t_x a_x = t_{xy}$$  \hspace{1cm} (22.1)

$$t_{xy} + t_{xy} a_x = b t_x + t_v^T (t_v^{-1})^{-1}$$  \hspace{1cm} (22.2)

$$t_v + t_v a_x = -t_{xy} t_x^{-1} t_v$$  \hspace{1cm} (22.3)

where $a_{x,v} = -a_{x,v}^T$ - antisymmetric matrices.

By using equations (22) it may be shown that product $T T^T$ satisfies to equation (5) for the matrix $M$ of the second order moments. Thereby the equality:

$$T T^T = M$$  \hspace{1cm} (23)

is valid at any point $s$ if it is valid at initial point of the focusing system.

Due to antisymmetry of matrix $A_1$ the transfer matrix of system $R$ (17) may be represent in the following form:

$$R = T Q_4 T_0^{-1},$$  \hspace{1cm} (24)

where $Q_4$ is orthogonal matrix of forth order, i.e.:

$$Q_4 Q_4^T = E_4$$  \hspace{1cm} (25)

The expression for matrix $Q_4$ may be found by using the new variables $W$:

$$Y_1 = Q_w W \quad Q_w = \begin{pmatrix} Q_x & 0 \\ 0 & Q_y \end{pmatrix},$$  \hspace{1cm} (26)

$Q_{x,v}$ - matrices of rotation diagonalizing matrix $t_x^{-1} t_v$:

$$Q_x^T t_x^{-1} t_v = \beta^{-1} = \begin{pmatrix} 1/ \beta_1 & 0 \\ 0 & 1/ \beta_2 \end{pmatrix}$$  \hspace{1cm} (27)

With these definitions vector $W$ satisfies to equation:

$$W' = A_w W \quad A_w = \begin{pmatrix} 0 & \beta^{-1} \\ -\beta^{-1} & 0 \end{pmatrix},$$  \hspace{1cm} (28)

if the antisymmetic matrices $a_{x,v}$ (21) is defined as:

$$a_{x,v} = Q_{x,v} Q_{x,v}^T$$  \hspace{1cm} (29)

The quantities $\beta_{1,2}$ coincide with the square root of the eigenvalues of matrix $B$:

$$B = M_{xx}^{1/2} (M_{yy} - M_{xy}^T M_{xx}^{-1} M_{xy})^{-1} M_{xx}^{1/2}$$  \hspace{1cm} (30)

and therefore is determined by the second order moments.
Diagonal form of matrix $\beta$ gives possibility to find transfer matrix $R_w$ for the phase space variable $W$:

$$W = R_w W_0 \quad R_w = \begin{pmatrix} C_w & S_w \\ -S_w & C_w \end{pmatrix}$$ \hfill (31.1)

$$C_w = \begin{pmatrix} \cos \mu_1 & 0 \\ 0 & \cos \mu_2 \end{pmatrix} \quad S_w = \begin{pmatrix} \sin \mu_1 & 0 \\ 0 & \sin \mu_2 \end{pmatrix}$$ \hfill (31.2)

Phase advances $\mu_{1,2}$ connects with functions $\beta_{1,2}$ (27):

$$\mu_i' = 1/\beta_i, \quad i = 1,2$$ \hfill (32)

As it follows from (31) pairs of phase space variables $(w_1, w_3)$ and $(w_2, w_4)$ form two independent 2D subspaces of the whole four-dimensional phase space.

By using expressions (31) the orthogonal matrix $Q_4$ (25) may be defined as

$$Q_4 = Q_0 R_w Q_0^T$$

and transfer matrix $R$ (17), (25) has the following form:

$$R = T Q_w R_w Q_w^T T^{-1}$$ \hfill (32)

**MOMENTUM SPREAD**

The momentum spread may be taking into account by introducing new phase space variable $Y_p = (x_1, x_2, x_1', x_2', \delta) = (Y^T, \delta)$, where $\delta = \Delta p / p$ is relative deviation of particle momentum from average value. Vector $Y_p$ satisfies to equation:

$$Y_p' = A_p Y_p \quad A_p = \begin{pmatrix} 0 & \Sigma \\ b_p & a_p \end{pmatrix} \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ \hfill (33)

Here $b_p$ is 3x2 rectangular matrix and $a_p$ is 3x3 matrix.

In this case we may use the system of coordinate (20) with changing of dimensions of matrix $T$ elements. Matrix $t_x$ has the same 2x2 order as in previous case, $t_{xy}$ is 3x2 rectangular matrix, and $t_y$ is 3x3 matrix. The equations for elements of matrix $T$ may be found by the same manner as in previous section:

$$t_x' + a_{xp} t_x = \Sigma t_{xy}$$ \hfill (33.1)

$$t_{xy}' + t_{xy} a_{xp} = b t_x + a t_{xy} + t_{xy} \Sigma t_x (t_x^T)^{-1}$$ \hfill (33.2)

$$t_y' + a_{yp} t_y = a t_y - t_{xy} t_{xy}^{-1} \Sigma t_y$$ \hfill (33.3)

where $a_{xp}$, $a_{yp}$ are 2x2 and 3x3 antisymmetric matrices correspondingly. As in the previous case matrix $T$ is connected with matrix $M$ of the second order moments by equality (23). With these definitions vector $Y_{ip} = T^{-1} Y_p$ satisfies the following equation:

$$Y_{ip}' = A_{1p} Y_{ip} \quad A_{1p} = \begin{pmatrix} a_{xp} & t_x^{-1} \Sigma t_y \\ -(t_x^{-1} \Sigma t_y)^T a_{yp} \end{pmatrix}$$ \hfill (34)

The transfer matrix $R_p$ has the same form as matrix $R$ defined by formula (24):

$$R_p = T Q_5 T_0^{-1}$$ \hfill (35)

where $Q_5$ is the orthogonal matrix of the fifth order. It may be found by the same manner as in the previous case:

$$Q_5 = Q_p R_{wp} Q_p^T \quad Q_p = \begin{pmatrix} Q_{xp} & 0 \\ 0 & Q_{yp} \end{pmatrix}$$ \hfill (36)

Here $Q_{xp}$ and $Q_{yp}$ are rotational matrices of the second and third order correspondingly giving the following result of the matrix $t_x \Sigma t_y$ transformation:

$$Q_{xp} \Sigma t_y Q_{yp}^{-1} = \begin{pmatrix} 1/\beta_1 & 0 \\ 0 & 1/\beta_2 \end{pmatrix}$$ \hfill (37)

The quantities $1/\beta_{1,2}$ coincides with the square root of the eugenvalues of matrix $B_p$ defined by the second order moments:

$$B_p = M_{xx}^{-1/2} \Sigma (M_{yy} - M_{xy}^2 M_{xx}) \Sigma^T M_{xx}^{-1/2}$$ \hfill (38)

Matrix $R_{wp}$ in (36) connects with matrix $R_w$ (31):

$$R_{wp} = \begin{pmatrix} R_w & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} C_w & -S_w & 0 \\ 0 & C_w & 0 \\ S_w & 0 & C_w \end{pmatrix}$$ \hfill (39)

The phase advances $\mu_{1,2}$ are defined by beta functions (37) with the help of expressions (32).

**REFERENCES**