NONLINEAR BEAM DYNAMICS AND EFFECTS OF WIGGLERS

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Abstract

We present the applications of variational–wavelet approach for the analytical/numerical treatment of the effects of insertion devices on beam dynamics. We investigate the dynamical models which have polynomial nonlinearities and variable coefficients. We construct the corresponding wavelet representation for wigglers and undulator magnets.

1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the treatment of effects of insertion devices on beam dynamics. Our approach in this paper is based on the generalization of variational-wavelet approach from [1]-[8], which allows us to consider not only polynomial but rational type of nonlinearities [9]. We present solution via full multiresolution expansion in all time scales, which gives us expansion into a slow part and fast oscillating parts. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. In this way we give contribution to our full solution from each scale of resolution or each time scale. The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution. Starting from formulation of initial dynamical problems (part 2) we construct in part 3 via multiresolution analysis explicit representation for all dynamical variables in the base of compactly supported wavelets. Then in part 4 we consider further extension of our previous results to the case of variable coefficients.

2 EFFECTS OF INSERTION DEVICES ON BEAM DYNAMICS

Assuming a sinusoidal field variation, we may consider according to [10] the analytical treatment of the effects of insertion devices on beam dynamics. One of the major detrimental aspects of the installation of insertion devices is the resulting reduction of dynamic aperture. Introduction of non-linearities leads to enhancement of the amplitude-dependent tune shifts and distortion of phase space. The nonlinear fields will produce significant effects at large betatron amplitudes such as excitation of n–order resonances. The components of the insertion device vector potential used for the derivation of equations of motion are as follows:

$$\begin{aligned} A_x &= \cosh(k_x x)\cosh(k_y y)\sin(ks)/(k\rho) \qquad (1)\\ A_y &= k_x \sinh(k_x x)\sinh(k_y y)\sin(ks)/(k_y k\rho) \end{aligned}$$

with $k_x^2+k_y^2=k^2=(2\pi/\lambda)^2$, where λ is the period length of the insertion device, ρ is the radius of the curvature in the field B_0. After a canonical transformation to betatron variables, the Hamiltonian is averaged over the period of the insertion device and hyperbolic functions are expanded to the fourth order in x and y (or arbitrary order). Then we have the following Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}[p_x^2+p_y^2]+\frac{1}{4k^2\rho^2}[k_x^2x^2+k_y^2y^2]\\ &+ \frac{1}{12k^2\rho^2}[k_x^4x^4+k_y^4y^4+3k_x^2k^2x^2y^2] \qquad (2)\\ &- \frac{\sin(ks)}{2k\rho}[p_x(k_x^2x^2+k_y^2y^2)-2k_x^2p_yxy] \end{aligned}$$

We have in this case also nonlinear (polynomial with degree 3) dynamical system with variable (periodic) coefficients. After averaging the motion over a magnetic period we have the following related equations

$$\begin{aligned} \ddot{x} &= -\frac{k_x^2}{2k^2\rho^2}\left[x+\frac{2}{3}k_x^2x^3\right]-\frac{k_x^2xy^2}{2\rho^2} \qquad (3)\\ \ddot{y} &= -\frac{k_y^2}{2k^2\rho^2}\left[y+\frac{2}{3}k_y^2y^3\right]-\frac{k_x^2x^2y}{2\rho^2} \end{aligned}$$

3 WAVELET FRAMEWORK

The first main part of our consideration is some variational approach to this problem, which reduces initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. Multiresolution expansion is the second main part of our construction. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of increasing closed subspaces V_j: $...V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\subset ....$ The solution is parameterized by solutions of two reduced algebraical problems, one is nonlinear and the second are some linear problems, which are obtained by the method of Connection Coefficients (CC)[11]. We use compactly supported wavelet basis. Let our wavelet expansion be

$$f(x)=\sum_{\ell\in\mathbf{Z}}c_\ell\varphi_\ell(x)+\sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}}c_{jk}\psi_{jk}(x) \qquad (4)$$

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If $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only $f(x) = \sum\limits_{\ell \in \mathbf{Z}} c_{J\ell}\varphi_{J\ell}(x)$. This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let be $\varphi^n_\ell = \mathrm{d}^n\varphi_\ell(x)/\mathrm{d}x^n$. We consider computation of the wavelet - Galerkin integrals. Let $f^d(x)$ be d-derivative of function $f(x)$, then we have $f^d(x) = \sum_\ell c_l\varphi^d_\ell(x)$, and values $\varphi^d_\ell(x)$ can be expanded in terms of $\varphi(x)$

$$\begin{aligned} \varphi^d_\ell(x) &= \sum_m \lambda_m\varphi_m(x), \\ \lambda_m &= \int\limits_{-\infty}^{\infty} \varphi^d_\ell(x)\varphi_m(x)\mathrm{d}x, \end{aligned} \tag{5}$$

where λ_m are wavelet-Galerkin integrals. The coefficients λ_m are 2-term connection coefficients. In general we need to find $(d_i \geq 0)$

$$\Lambda^{d_1 d_2 \dots d_n}_{\ell_1 \ell_2 \dots \ell_n} = \int\limits_{-\infty}^{\infty} \prod \varphi^{d_i}_{\ell_i}(x)dx \tag{6}$$

For Riccati case we need to evaluate two and three connection coefficients

$$\begin{aligned} \Lambda^{d_1 d_2}_\ell &= \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi^{d_2}_\ell(x)dx, \\ \Lambda^{d_1 d_2 d_3} &= \int\limits_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi^{d_2}_\ell(x)\varphi^{d_3}_m(x)dx \end{aligned} \tag{7}$$

According to CC method [11] we use the next construction. When N in scaling equation is a finite even positive integer the function $\varphi(x)$ has compact support contained in $[0, N-1]$. For a fixed triple (d_1, d_2, d_3) only some $\Lambda^{d_1 d_2 d_3}_{\ell m}$ are nonzero: $2 - N \leq \ell \leq N-2, \quad 2-N \leq m \leq N-2, \quad |\ell - m| \leq N-2$. There are $M = 3N^2 - 9N + 7$ such pairs (ℓ, m). Let $\Lambda^{d_1 d_2 d_3}$ be an M-vector, whose components are numbers $\Lambda^{d_1 d_2 d_3}_{\ell m}$. Then we have the first reduced algebraical system : Λ satisfy the system of equations $(d = d_1 + d_2 + d_3)$

$$\begin{aligned} A\Lambda^{d_1 d_2 d_3} &= 2^{1-d}\Lambda^{d_1 d_2 d_3}, \\ A_{\ell,m;q,r} &= \sum_p a_p a_{q-2\ell+p} a_{r-2m+p} \end{aligned} \tag{8}$$

By moment equations we have created a system of $M + d + 1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second reduced algebraical system gives us the 2-term connection coefficients ($d = d_1 + d_2$):

$$A\Lambda^{d_1 d_2} = 2^{1-d}\Lambda^{d_1 d_2}, \quad A_{\ell,q} = \sum_p a_p a_{q-2\ell+p} \tag{9}$$

For nonquadratic case we have analogously additional linear problems for objects (6). Solving these linear problems we obtain the coefficients of reduced nonlinear algebraical system and after that we obtain the coefficients of wavelet expansion (4). As a result we obtained the explicit time solution of our problem in the base of compactly supported wavelets. On Fig.1 we present an example of basis wavelet function which satisfies some boundary conditions. In the following we consider extension of this approach to the case of arbitrary variable coefficients.

4 VARIABLE COEFFICIENTS

In the case when we have the situation when our problems (2),(3) are described by a system of nonlinear (rational) differential equations, we need to consider also the extension of our previous approach which can take into account any type of variable coefficients (periodic, regular or singular). We can produce such approach if we add in our construction additional refinement equation, which encoded all information about variable coefficients [12]. According to our variational approach we need to compute only additional integrals of the form

$$\int_D b_{ij}(t)(\varphi_1)^{d_1}(2^m t - k_1)(\varphi_2)^{d_2}(2^m t - k_2)\mathrm{d}x, \tag{10}$$

where $b_{ij}(t)$ are arbitrary functions of time and trial functions φ_1, φ_2 satisfy the refinement equations:

$$\varphi_i(t) = \sum_{k \in \mathbf{Z}} a_{ik}\varphi_i(2t - k) \tag{11}$$

If we consider all computations in the class of compactly supported wavelets then only a finite number of coefficients do not vanish. To approximate the non-constant coefficients, we need choose a different refinable function φ_3 along with some local approximation scheme

$$(B_\ell f)(x) := \sum_{\alpha \in \mathbf{Z}} F_{\ell,k}(f)\varphi_3(2^\ell t - k), \tag{12}$$

where $F_{\ell,k}$ are suitable functionals supported in a small neighborhood of $2^{-\ell}k$ and then replace b_{ij} in (10) by $B_\ell b_{ij}(t)$. In particular case one can take a characteristic function and can thus approximate non-smooth coefficients locally. To guarantee sufficient accuracy of the resulting approximation to (10) it is important to have the flexibility of choosing φ_3 different from φ_1, φ_2. In the case when D is some domain, we can write

$$b_{ij}(t) \mid_D = \sum_{0 \leq k \leq 2^\ell} b_{ij}(t)\chi_D(2^\ell t - k), \tag{13}$$

where χ_D is characteristic function of D. So, if we take $\varphi_4 = \chi_D$, which is again a refinable function, then the problem of computation of (10) is reduced to the problem of calculation of integral

$$\begin{aligned} H(k_1, k_2, k_3, k_4) = H(k) = \int_{\mathbf{R}^s} \varphi_4(2^j t - k_1) \cdot \\ \varphi_3(2^\ell t - k_2)\varphi_1^{d_1}(2^r t - k_3)\varphi_2^{d_2}(2^s t - k_4)\mathrm{d}x \end{aligned} \tag{14}$$

The key point is that these integrals also satisfy some sort of refinement equation [12]:

$$2^{-|\mu|}H(k) = \sum_{\ell \in \mathbf{Z}} b_{2k-\ell} H(\ell), \qquad \mu = d_1 + d_2. \quad (15)$$

This equation can be interpreted as the problem of computing an eigenvector. Thus, we reduced the problem of extension of our method to the case of variable coefficients to the same standard algebraical problem as in the preceding sections. So, the general scheme is the same one and we have only one more additional linear algebraic problem by which we can parameterize the solutions of corresponding problem in the same way.

On Fig. 2 we present approximated configuration and on Fig. 3 the corresponding multiresolution representation according to formula (4).

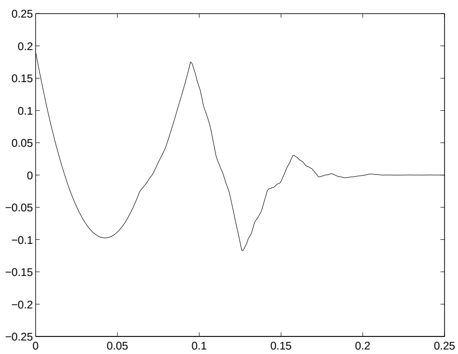

Figure 1: Basis wavelet with fixed boundary conditions

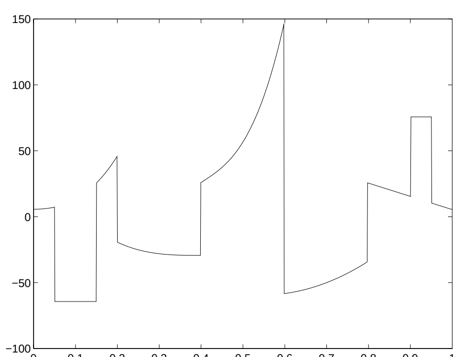

Figure 2: Approximated configuration

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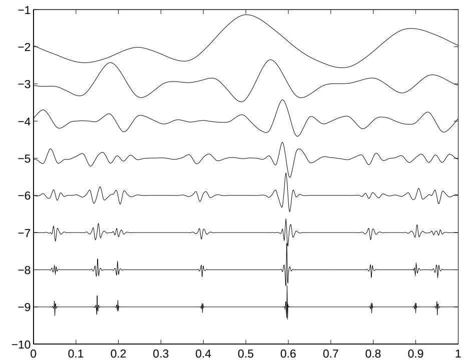

Figure 3: Multiresolution representation

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