Abstract
A semi-numerical method of integrating the Vlasov equation to obtain beam transfer functions directly as a function of frequency is presented. The results are compared with beam transfer functions calculated via particle tracking and excellent agreement is shown. The technique works well with both transverse wakes and detuning wakes from space charge.

INTRODUCTION AND THEORY
The stability and response properties of particle beams are encapsulated in beam transfer functions (BTFs) [1, 2, 3]. BTFs are relatively easy to measure making their accurate calculation of great practical interest. In this paper we will present two independent ways to calculate bunched beam transverse beam transfer functions, and show their agreement for some challenging parameters. To begin let $\theta$ denote azimuth, which increases by $2\pi$ each turn. Let $t$ be clock time and let $\tau$ denote particle arrival time with respect to the synchronous particle so that $\theta = \omega_0(t - \tau)$, with $\omega_0$ the synchronous angular revolution frequency. We use $\theta$ as the time-like variable and, to make a precise cross check possible, take linear rf so that

$$\frac{d\tau}{d\theta} = Q_s \epsilon, \quad \frac{d\epsilon}{d\theta} = -Q_s \tau,$$  \hspace{1cm} (1)

where $Q_s$ is the synchrotron tune and $\epsilon$ is the energy variable. With $\epsilon = 0$ we consider a single transverse variable $x$ and uniform focusing. Collective forces are taken in the continuum approximation with $dx/d\theta = p$ and

$$\frac{dp}{d\theta} = -Q_0^2(\epsilon)x + F_c(\theta, \tau) + 2Q_0\Delta Q_{sc}(\tau)[x - \bar{x}(\theta, \tau)]$$

$$- \frac{q}{2\pi P_0\omega_0} \int_0^{2\tau_0} W_x(\tau_1)D(\theta, \tau - \tau_1) d\tau_1,$$  \hspace{1cm} (2)

where $Q_0(\epsilon) = Q_0 + Q_s\epsilon \omega_0\xi/\eta$ is the betatron tune, with chromaticity $\xi$ and frequency slip factor $\eta$, $F_c(\theta, \tau)$ is the external driving force, $\Delta Q_{sc}(\tau)$ is the space charge tune shift as a function of longitudinal position in the bunch, and $\bar{x}(\theta, \tau)$ is the average beam offset. The wall induced forces are due to the transverse wake potential $W_x(\tau)$ and driven by the instantaneous dipole moment

$$D(\theta, \tau) = \bar{x}(\theta, \tau)I(\tau)$$

with $I(\tau)$ the bunch current, $q$ the particle charge, and $F_0$ the synchronous momentum. The force is limited to a single bunch of full length $2\tau_0$ but can be extended to uniformly filled rings.

To solve (1) and (2) using the Vlasov equation first introduce amplitude angle variables defined by $\tau = a \sin \psi$, $\epsilon = a \cos \psi$. Since all particle have constant $a$,

$$\frac{\partial F}{\partial \theta} + p \frac{\partial F}{\partial x} + \frac{dp}{d\theta} \frac{\partial F}{\partial p} + Q_s \frac{\partial F}{\partial \psi} = 0,$$  \hspace{1cm} (3)

where $dp/d\theta$ is given by Eq (2). We normalise $F$ so that $F dx dp d\theta d\psi$ is the number of particles in the phase space volume. To continue we define 3 transverse moments [4].

$$(X(\psi, a, \theta), P, \Psi) = \int dx dp F(x, p, a, \psi, \theta) \{x, p, 1\}.$$  \hspace{1cm} (4)

This gives

$$\frac{\partial X}{\partial \theta} + Q_s \frac{\partial X}{\partial \psi} = P(\psi, a, \theta),$$  \hspace{1cm} (5)

$$\frac{\partial P}{\partial \theta} + Q_s \frac{\partial P}{\partial \psi} = -Q_0^2(\epsilon)X + F_c(\theta, \tau)\Psi$$

$$+ 2Q_0\kappa [\int \partial d\epsilon \Psi - \Psi \int \partial dX]$$

$$- \frac{q}{2\pi P_0\omega_0} \int_0^{2\tau_0} W_x(\tau_1)q \int d\epsilon X(\epsilon, \tau - \tau_1)$$  \hspace{1cm} (6)

where $\Delta Q_{sc}(\tau) = \kappa \int d\epsilon \Psi, \bar{x}(\tau) \int d\epsilon \Psi = \int dX$ and, for example,

$$\int dX = \int a_1 da_1 d\psi X(a_1 \sin \psi - a_1 \sin \psi_1)X(\psi_1, a_1, \theta),$$

and the occurrences of $\tau$ and $\epsilon$ in (6) are understood to be shorthand for the amplitude angle representations. Next we substitute (5) in (6) and drop second partial derivatives with respect to $\psi$. We take $F_c(\theta, \tau) = F_0(\tau) \exp(-iQ\theta)$, and we take $X = X_1 \exp(-iQ\theta - i\xi \omega_0 \tau/\eta)$, with $Q = Q_0 + \Delta Q$. This yields

$$\frac{\Delta Q + iQ_0 \frac{\partial}{\partial \psi} + \Delta Q_{sc}(a \sin \psi)}{2Q_0} X_1(\psi, a) = \tilde{F}(\tau)\Psi(a),$$  \hspace{1cm} (7)

where

$$\tilde{F}(\tau) = -\frac{e^{i\xi \omega_0 \tau/\eta}F_0(\tau)}{2Q_0} + \kappa \int dX_1$$

$$+ q^2 \int_0^{2\tau_0} W_x(\tau_1)e^{i\xi \omega_0 \tau_1/\eta} \int d\epsilon X_1(\epsilon, \tau - \tau_1).$$  \hspace{1cm} (8)
The strategy is to solve (7) for $D_1 = q \int d\tau X_1$ resulting in a one dimensional integral equation for $D_1$ which will be solved numerically. To proceed let $\Delta Q_{sc}(a \sin \psi) = \Lambda(a) + d\Lambda/d\psi$ where $\Lambda(a, \psi) = \Lambda(a, \psi + 2\pi)$. Substitute $X_1 = X_2 \exp(i\Lambda/Q_s)$ yielding

$$\left\{ \Delta Q + iQ_s \frac{\partial}{\partial \psi} + \lambda \right\} X_2 = e^{-i\Lambda/Q_s} \tilde{F} \Psi. \quad (9)$$

Multiply both sides by the integrating factor $\exp(-i\psi(\lambda + \Delta Q)/Q_s)$ and integrate from $\psi$ to $\psi + 2\pi$ employing the periodicity of $X_2$. Backsubstitute $X_1$ giving

$$X_1 = \frac{\Psi e^{iG(a, \psi)}}{Q_s R(a)} = \int_0^{\psi} d\psi_1 \tilde{F}(a \sin \psi_1)e^{-iG(a, \psi_1)}$$

where $R(a) = 1 - \exp(-2i\pi(\lambda + \Delta Q)/Q_s)$ and

$$G(\tau, \psi) = \int_0^{\psi} d\psi_1 \frac{\Delta Q + \Delta Q_{sc}(a \sin \psi_1)}{Q_s}.$$ 

To proceed note that

$$D_1(\tau) = q \int d\tau d\psi \delta(\tau - a \sin \psi) X_1(a, \psi)$$

yielding

$$D_1(\tau) = \int \tilde{G}(\tau, \tau_1) q \tilde{F}(\tau_1) d\tau_1 \quad (11)$$

where

$$q F(\tau) = \frac{- e^{i\xi_0 \tau/\eta} q F_0(\tau)}{2Q_0} + \kappa D_1(\tau)$$

$$+ \frac{q^2}{4\pi Q_0 \omega_0} \int_0^{\infty} d\tau_1 W(\tau_1) e^{i\xi_0 \tau_1/\eta} D_1(\tau - \tau_1) \quad (12)$$

and

$$\tilde{G}(\tau, \tau_1) = \int_0^{\infty} d\tau d\psi \delta(\tau - a \sin \psi) \Psi(a) e^{iG(a, \psi)}$$

$$K(a) \int_0^{\psi} d\psi_1 \delta(\tau_1 - a \sin \psi_1) e^{-iG(a, \psi_1)} \quad (13)$$

where $K = i\Psi/Q_s R$. To proceed assume $D_1(\tau)$ is defined at a set of equidistant points $\tau_k = k\Delta$ and that $D$ varies linearly between these points. Then

$$q \tilde{F}(\tau) = \sum_{k=-N}^{N} a_k T(\tau - k\Delta),$$

where $T(x)$ is a triangle function of height 1 and half width at base $\Delta$. For smooth bunches $D(\tau)$ is zero at the ends of the bunch so we take $\Delta = \tau_0/(N + 1)$ where $\tau_0$ is the half bunch length. Next we do the integral over $\psi$ in (13). For any smooth function $h(a, \psi)$,

$$\int_0^{2\pi} \delta(\tau - a \sin \psi) h(a, \psi) d\psi = \sum_{p=1,2} \frac{h(a, \psi_p)}{\sqrt{a^2 - \tau^2}}$$

where the two angles satisfy $\sin \psi_{1,2} = \tau/a$ and the integral vanishes for $|\tau| > \Delta$. Next we define a new radial variable $u$ defined by $a^2 = a^2 - \tau^2$ so $du = a da$. The dipole density is defined at the same set of lattice points as $\tilde{F}$ so

$$D_1(m\Delta) \equiv D_m = \sum_{n=-N}^{N} a_n M_{m,n}, \quad (14)$$

where

$$M_{m,n} = \frac{\sqrt{\tau_0 - m^2\Delta^2}}{2Q_0} \int_0^{\psi_p+2\pi} d\psi_1 T(n\Delta - a \sin \psi_1) e^{-iG(a, \psi_1)}, \quad (15)$$

In integral (15) $a = \sqrt{u^2 + m^2\Delta^2}$ and $\sin \psi_p = m\Delta/a$. We have written a program which solves (14) with matrix element (15) and $a_n$ from (12),

$$a_n = - \frac{e^{i\xi_0 \Delta/\eta} q F_0(n\Delta)}{2Q_0} + \kappa D_n$$

$$+ \frac{q^2}{4\pi Q_0 \omega_0} \sum_{k=0}^{2N+1} \Delta W(\tau_k) e^{i\xi_0 k\Delta/\eta} D_{n-k}. \quad (16)$$

SIMULATIONS

The solution of Equations (1) and (2) using particle tracking is straightforward [5, 6, 7, 8]. Before considering a sinusoidally drive BTF we consider forces of the form $F_0(\tau, \theta) = f(\tau) \cos(\Omega \theta) \exp(\delta_0 \theta)$ where $\delta_0$ is the periodic delta function, $\Omega$ is the real part of the drive tune, $g$ is the imaginary part of the drive tune and $f(\tau)$ is non-zero only over the full bunch length, $2\tau_b$. Suppose $D_f(\theta, \tau)$ is the dipole response to a kick $f(\tau)$ given at $\theta = 0$. Then the response from all the kicks is

$$D(\theta, \tau) = \sum_{m=-\infty}^{\infty} D_f(\theta - 2\pi m, \tau) \cos(2\pi m Q) \exp(2\pi gm). \quad (17)$$

To obtain a scalar we take the inner product

$$\tilde{D}(\theta) = \tau_b \int_{-\tau_b}^{\tau_b} d\tau f(\tau) D(\theta, \tau)$$

$$= \sum_{m=-\infty}^{\infty} \tilde{D}_f(\theta - 2\pi m) \cos(2\pi m Q) \exp(2\pi gm)$$

For a pickup at a fixed location one obtains the time series $\tilde{D}(0), \tilde{D}(2\pi), \tilde{D}(4\pi), \ldots$, these values may be obtained with a single simulation. As long as $\tilde{D}_f(\theta) = 0$ grows more quickly than $\exp(g\theta)$ we will have

$$\tilde{D}(2\pi k) = [A_Q \cos(2\pi kQ) + B_Q \sin(2\pi kQ)] \exp(2\pi gk)$$

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with

\[ A_Q + jB_Q = \sum_{\ell=0}^{\infty} \hat{B}_f(2\pi\ell) \exp(2\pi\ell[jQ - g]) \quad (18) \]

Hence, the entire BTF can be obtained with one simulation. For a real beam transfer function the force at the pickup is sinusoidal. We assume the frequency is \( n\omega_0 + \omega_1 \) where \( n \) is an integer, \(|\omega_1| \leq \omega_0/2\). Then the driving force is given by

\[
F_\epsilon(\theta, \tau) = \delta_p(\theta) \cos((n\omega_0 + \omega_1)\tau),
\]

\[
= \delta_p(\theta) \cos((n\omega_0 + \omega_1)(\tau + \theta/\omega_0)),
\]

\[
\approx \delta_p(\theta) \cos(n\omega_0\tau) \cos(\omega_1 k(t)T_0) - \delta_p(\theta) \sin(n\omega_0\tau) \sin(\omega_1 k(t)T_0) \quad (19)
\]

where \( T_0 = 2\pi/\omega_0 \) and \( k(t) = \text{nint}(t/T_0) \). In a real BTF only the response at the drive frequency is measured so, with a sinusoidal kick one needs to run two simulations. One with a kick proportional to \( \sin(n\omega_0\tau) \) and another with kick \( \cos(n\omega_0\tau) \). During each simulation one takes inner products of the response with each sinusoidal every turn. One finds

\[
BTF(\omega_1) = \sum_{m=0}^{\infty} (C_m + jS_m)e^{-j(\omega_1 + \epsilon)mT_0}
\]

with

\[
C_m = \int_{-\tau_n}^{\tau_n} d\tau D_{\cos}(2\pi m, \tau)e^{-jn\omega_0\tau},
\]

where \( D_{\cos}(\theta, \tau) \) is the response to a cosine kick at \( \theta = 0 \), and similarly for \( S_m \). In these equations we have used the electrical engineering convention with \( j = -i \) and allowed for an exponentially growing drive \( \propto \exp(\epsilon t) \). The driving terms in the Vlasov approach are given by \( F_0(\tau) = \exp(\pm jn\omega_0\tau) \) for the \( \pm n \) sidebands, respectively. Figure 1 shows upper and lower sideband BTFs with \( n\omega_0 = \pm \pi/2\tau_n \), varying chromaticity and a step function wake of size comparable to what is needed for a mode coupling instability. The peak space charge tune shift is 4 times the synchrotron tune. For more extreme parameters the disagreement increases and we continue to look for errors.

We would like to thank NERSC supported by the DOE under contract no. DE-AC02-05CH11231

REFERENCES


Figure 1: Simulated BTFs (solid lines) and those following from Eq (14) (crosses).