CSR IMPEDANCE FOR NON-ULTRARELATIVISTIC BEAMS*

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Abstract

For the analysis of the coherent synchrotron radiation (CSR) induced microbunching gain in the low energy regime, such as when a high-brightness electron beam is transported through a low-energy merger in an energy-recovery linac (ERL) design, it is necessary to extend the existing CSR impedance expression in the ultrarelativistic limit to the non-ultrarelativistic regime. This paper presents our analysis of CSR impedance for general beam energies.

INTRODUCTION

Modern accelerator designs often demand the generation and transport of high brightness electron beams. For these designs it is important to have accurate estimation of the coherent synchrotron radiation (CSR) effects on the degradation of the beam phase space quality. The analytical expressions of CSR wakefield are often utilized in time-domain particle tracking. For example, in ELEGANT, CSR effects are modelled for ultrarelativistic beams using CSR wakefield obtained for the steady-state interaction [1] or for the transient-state interaction [2]. On the other hand, the analytical expression of CSR impedance is necessary for the frequency-domain analysis, such as for the Vlasov anslysis of the microbunching gain [3,4]. For ultrarelativistic bunch in free space, the CSR impedance is given by [3,4]

$$Z(k) = -iA\, k^{1/3} R^{-2/3} \tag{1}$$

For

$$A = -2\pi[Bi'(0)/3 + iAi'(0)] = -0.94 + 1.63i \tag{2}$$

where Ai and Bi are Airy functions.

The designs of low-energy mergers in ERLs requires the knowledge of CSR interaction at low energy and also LSC interaction on a curved orbit. For time-domain particle tracking with codes such as GPT [5] or TStep [6], the study of CSR wakefields are extended from ultrarelativistic regime [1,2] to the low energy regime [7,8]. Similarly, to apply the Vlasov analysis of microbunching gain [3,4] in the frequency domain for the low energy regime, we need to extend the CSR impedance in Eq. (1) to the low energy regime. In the following we present our analysis of the steady-state CSR impedance for general beam energies. The impedance expression reduces to Eq. (1) under ultrarelativistic approximation. In addition, it is shown that the real part of the CSR impedance is consistent with the synchrotron-radiation power loss spectrum given by Schwinger [9]. Note that an existing expression of CSR impedance in free space for general beam energies was presented earlier [10], which takes different form from our expression. Relation between the two will be established in our coming-up studies.

ANALYSIS OF CSR IMPEDANCE FOR GENERAL BEAM ENERGIES

Consider a rigid line bunch, with the longitudinal density distribution $\lambda(z)$, moving at velocity v on a circular orbit with radius R. We will start with the longitudinal wakefield on the bunch as a result of steady-state CSR interaction in free space, and obtain the impedance from the Fourier transform of the CSR wakefield.

First, the electric field on a particle at (s,t), due to the CSR interaction from all other particles in the bunch, is expressed in terms of the retarded potentials

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{c\,\partial t},$$

$$\Phi(s,t) = \int_{-\infty}^{\infty} ds' \frac{\rho(s',t')}{|\vec{r}(s)-\vec{r}(s')|}, \quad \vec{A}(s,t) = \int_{-\infty}^{\infty} ds' \frac{\vec{\beta}(t')\rho(s',t')}{|\vec{r}(s)-\vec{r}(s')|}.$$

Here the retarded time is

$$t' = t - |\vec{r}(s)-\vec{r}(s')|/c$$

and the longitudinal charge density distribution is

$$\rho(s,t) = e\lambda(z) \quad \text{for} \quad z = s - \beta ct.$$

The energy loss rate for the particle per unit path length is

$$e\hat{\beta}\cdot\vec{E} = -e\frac{d\Phi}{vdt} + e\left(\frac{\partial\Phi}{v\partial t} - \vec{\beta}\cdot\frac{\partial\vec{A}}{v\partial t}\right).$$

For the rigid line bunch on the circular orbit, we have

$$\vec{\beta} = (\beta_r, \beta_\theta, \beta_z) = (0, v/c, 0) \quad \text{and} \quad \frac{d\Phi}{dt} = 0.$$

The longitudinal wakefield on the particle is subsequently

$$E_\theta = -\left(\frac{\partial\Phi}{\partial z} - \vec{\beta}\cdot\frac{\partial\vec{A}}{\partial z}\right),$$

in which

$$\Phi - \vec{\beta}\cdot\vec{A} = e\int_{-\infty}^{\infty} ds' \frac{(1-\beta^2) + \beta^2\left[1-\cos\left((s-s')/R\right)\right]}{|\vec{r}(s)-\vec{r}(s')|}\lambda(z')$$

for $z' = z - (s-s') + \beta|\vec{r}(s)-\vec{r}(s')|$.

Using Fourier expansion,

$$\lambda(z) = \int_{-\infty}^{\infty} dk\,\lambda(k)e^{ikz}, \quad E_\theta(z) = -e\int_{-\infty}^{\infty} dk\; Z(k)\lambda(k)e^{ikz}, \tag{3}$$

we obtain the CSR impedance

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$$Z(k)=ik\int_{-\infty}^{\infty}ds'g(s,s')\,e^{-ik\left((s-s')-\beta\left|2R\sin[(s-s')/R]\right|\right)}$$

where

$$g(s,s')=\frac{\gamma^{-2}+\beta^2\left(1-\cos\left[(s-s')/R\right]\right)}{\left|2R\sin\left[(s-s')/R\right]\right|}.$$

Next, with the change of the variables

$$\Delta s=\begin{cases}s-s' & \text{for } s'<s\\ s'-s & \text{for } s'>s\end{cases},$$

the impedance becomes

$$Z(k)=ik\int_0^{\infty}d\Delta s\; g(\Delta s)\times\left(e^{-ik\left[\Delta s-\beta 2R\sin(\Delta s/2R)\right]}+e^{-ik\left[-\Delta s-\beta 2R\sin(\Delta s/2R)\right]}\right).\quad(4)$$

Here the first exponential term represents contribution of the tail-head (*s'<s*) interaction, while the second one accounts for the head-tail (*s<s'*) interaction. Remark that the impedance in Eq. (4) contains contributions from both the CSR wakefield and the longitudinal space charge wakefield. The net power loss of the particles corresponds to the real part of the CSR impedance, while the longitudinal space charge interaction is reactive and only shows up in the imaginary part of the impedance.

For Δs < R, we get from Eq. (4) the expression for the real part of CSR impedance

$$\mathrm{Re}\left[Z(k)\right]=k\int_0^{\infty}d\Delta s\frac{\gamma^{-2}+(\Delta s)^2/(2R^2)}{\Delta s}\left(\sin\left[k\left(\frac{\Delta s}{2\gamma^2}+\frac{(\Delta s)^3}{24R^2}\right)\right]-\sin(2k\Delta s)\right).\quad(5)$$

This is compared with the incoherent synchrotron radiation power given by Eq. (II.5) of Ref. [9] for $\omega=k\nu$,

$$P(\omega)=\frac{e^2}{\pi}\frac{\omega}{c}\int_0^{\infty}\left(\gamma^{-2}+\frac{1}{2}\frac{c^2\tau^2}{R^2}\right)\times\left\{\sin\left((\omega(1-\beta)\tau+\frac{\omega c^2}{24R^2}\tau^3\right)\frac{d\tau}{\tau}-\sin\left(2\omega\tau\right)\frac{d\tau}{\tau}\right\}$$

or

$$P(\omega)=\frac{e^2}{\pi}\frac{k}{\gamma^2\sqrt{3}}\int_{2\omega/\omega_c}^{\infty}K_{5/3}(\eta)d\eta\quad(\text{for }\ \omega_c=3\omega_0\gamma^3)\quad(6)$$

Note that Eqs. (5) and (6) are related by

$$P(\omega)=\frac{1}{2\pi}e^2\left[2\,\mathrm{Re}\left(Z(k)\right)\right].\quad(7)$$

with the factor $(2\pi)^{-1}$ in Eq. (7) caused by the difference in the definition of Fourier transform between that used in Ref. [9] and our formula in Eq. (3). We then have

$$\mathrm{Re}\left[Z(k)\right]=\frac{k}{\gamma^2\sqrt{3}}\int_{2\omega/\omega_c}^{\infty}K_{5/3}(x)dx\quad\text{for}\quad\frac{\omega}{\omega_c}=\frac{kR}{3\gamma^3}.$$

With the equalities for modified Bessel functions [9] and for the Airy functions [11]

$$\int_x^{\infty}K_{5/3}(\eta)\,d\eta=2K_3(x)+\int_0^x K_{1/3}(\eta)d\eta-\frac{\pi}{\sqrt{3}}$$

and

$$K_{2/3}(x)=-\pi\frac{\sqrt{3}}{\zeta}\mathrm{Ai}'(\zeta),\ K_{1/3}(x)=\pi\sqrt{\frac{3}{\zeta}}\mathrm{Ai}(\zeta)\quad\text{for }\zeta=\left(\frac{3x}{2}\right)^{2/3},$$

the real part of the CSR impedance is reduced to (for $\mu=\gamma^{-2}\left(kR\right)^{2/3}$)

$$\mathrm{Re}\left[Z_{CSR}(k)\right]=\frac{k^{1/3}}{R^{2/3}}(-2\pi)\mathrm{Ai}'\left(\mu\right)+\frac{k\pi}{\gamma^2}\left(\int_0^{\mu}\mathrm{Ai}(\zeta)\mathrm{d}\zeta-\frac{1}{3}\right).\quad(8)$$

At ultrarelativistic limit $\gamma\to\infty$, Eq. (8) is reduced to the previous results in Eqs. (1) and (2):

$$\mathrm{Re}[Z(k)\xrightarrow{v=c}\frac{k^{1/3}}{R^{2/3}}(-2\pi)\mathrm{Ai}'\left(0\right)\quad(9)$$

Similarly, the imaginary part of CSR impedance can be obtained from Eq. (4)

$$\mathrm{Im}\left[Z(k)\right]=k\int_0^{\infty}d\Delta s\frac{\gamma^{-2}+(\Delta s)^2/(2R^2)}{\Delta s}\left(\cos\left[k\left(\frac{\Delta s}{2\gamma^2}+\frac{(\Delta s)^3}{24R^2}\right)\right]+\cos(2k\Delta s)\right).\quad(10)$$

Note that unlike the real part of impedance, for which the integrand in Eq. (5) does not have singularity, here the imaginary impedance contains terms with singular integrand. It is understood that the singularity reflects local space-charge interaction. However, since on a curved orbit pairwise particle interaction involves retardation, so the longitudinal space charge force will behave differently from that on a straight section. Just as the analysis of LSC on a straight path, the physical meaningful results for LSC force on a curved orbit requires us to take into account the 3D bunch distribution, which is beyond the scope of this paper. In this paper, our focus is on the CSR impedance and thus we only take into account the non-singular part of the imaginary impedance:

$$\mathrm{Im}\left[Z(k)\right]=k\int_0^{\infty}d\Delta s\frac{\Delta s}{2R^2}\cos\left[k\left(\frac{\Delta s}{2\gamma^2}+\frac{(\Delta s)^3}{24R^2}\right)\right].\quad(11)$$

Let $\Delta s=2t(R^2/k)^{1/3}$, $\mu=\gamma^{-2}\left(kR\right)^{2/3}$, and use the equality in Ref. [11], one gets

$$\mathrm{Im}[Z_{CSR}(k)]\simeq\frac{k^{1/3}}{R^{2/3}}\left\{\frac{2\pi}{3}\mathrm{Bi}'(\mu)+2\pi\int_0^{\mu}\left[Ai'(\mu)\mathrm{Bi}(\varsigma)-Ai(\varsigma)\mathrm{Bi}'(\mu)\right]d\varsigma\right\}.\quad(12)$$

At ultrarelativistic limit, $\mu\to 0$, we have

$$\mathrm{Im}[Z_{CSR}(k)]\xrightarrow{v=c}\frac{k^{1/3}}{R^{2/3}}\frac{2\pi}{3}Bi'\left(0\right),$$

which agrees with the existing results in Eqs. (1) and (2). Note that for obtaining the curvature effect on the LSC impedance, one can subtract [2] the straight-path LSC impedance from that of Eq. (10).

NUMERICAL ILLUSTRATION

In this section, we first illustrate the dependence of the non-ultrarelativistic CSR impedance in Eqs. (8) and (12) on beam energy for a given frequency, and compare the results with that of the ultrarelativistic case [Eq. (1)]. Then, we plot the impedance spectrum at given low energy. Finally we calculate the wakefield for a 1D Gaussian bunch using the impedance for non-ultrarelativistic beam, and demonstrate that our results agree well with the existing results [8], obtained directly from time-domain analysis.

In Fig. 1 we show the energy dependence of CSR impedances for non-ultrarelativistic beams. It can be clearly seen that at low beam energy, e.g. 10 MeV ($\gamma \approx 20$), both the real and imaginary parts of the CSR impedance deviate considerably from those of ultrarelativistic case. In the specific case with $\lambda \approx 200$ µm, it can be observed that the ultra-relativistic expression is valid only when $\gamma > 200$ (or, equivalently, $kR/\gamma^3 \ll 1$). Furthermore, the shorter the modulation wavelength (or, the larger the wave number) is, the higher the beam energy is required for keeping the validity of Eq. (1) in ultra-relativistic regime.

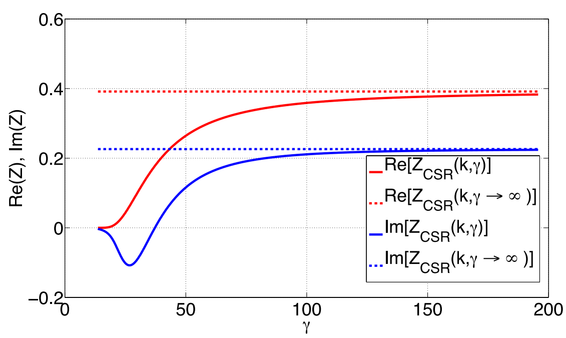

Figure 1: Energy dependence of the CSR impedance [Eqs.(8) and (12)]. Here we assume $k \approx 314$ cm^{-1} (or, $\lambda \approx 200$ µm).

For the example with $E = 10$ MeV and $R = 1.5$ m, Fig. 2 illustrates the real and imaginary parts of the CSR impedances. It is shown that the general impedance results agree with the ultrarelativistic ones only when $k/k_c << 1$.

In Fig. 3 we show the wakefield calculated by Eq. (3) for a Gaussian bunch distribution and the CSR impedance given by Eqs. (8) and (12). It can be seen that for a low energy beam, for example $E = 10$ MeV, the CSR wake could deviate considerably from that of the ultrarelativistic case, with the latter being an overestimation of the former one. Here the behaviours of wakefields for various beam energies agree well with that presented in Fig. 4(d) of Ref. [8].

SUMMARY

In this paper, we have extended the existing analytical formula for the free-space steady-state CSR impedance, given by Eqs. (1) and (2) for ultrarelativistic regime, to the low beam energy regime as summarized by Eqs. (8) and (12). This is particular useful for the microbunching studies in a low energy merger in ERL designs, for which the CSR impedance at low energy is required in the Vlasov solver. Our result is consistent with the radiation

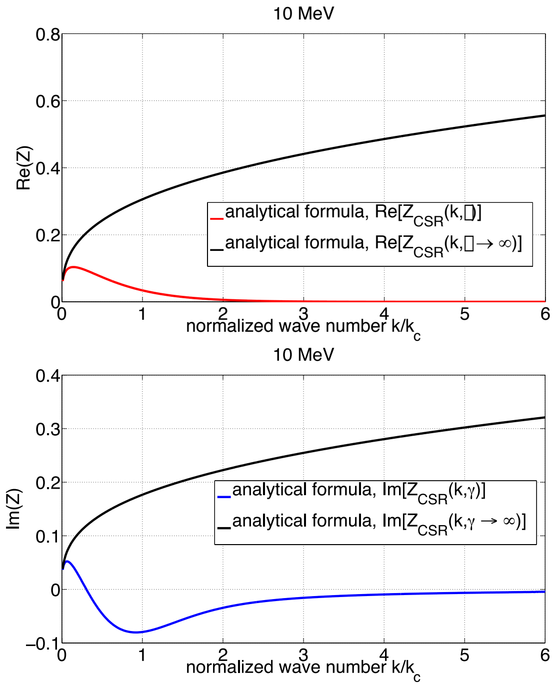

Figure 2: Real and imaginary parts of CSR impedance [Eqs. (8) and (12)]. Here $E = 10$ MeV and $R = 1.5$ m.

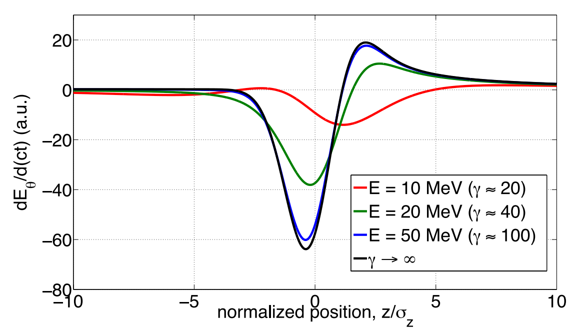

Figure 3: CSR wakefields with Gaussian bunch distribution for several different beam energies.

power spectrum derived by Schwinger [9], and agrees with the analytical expression of CSR impedance in the ultrarelativistic regime. We also numerically illustrate the energy and frequency dependences of the CSR impedance, and further reproduce the steady-state wakefields at different energies for a Gaussian bunch.

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