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CHROMATICITY & DISPERSION IN NONLINEAR INTEGRABLE OPTICS*

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Abstract

Proton accumulator rings and other circular hadron accelerators are susceptible to intensity- driven parametric instabilities because the zero-current charged particle dynamics are characterized by a single tune. Landau damping can suppress these instabilities, which requires energy spread in the beam or introducing nonlinear magnets such as octupoles. However, this approach reduces dynamic aperture. Nonlinear integrable optics can suppress parametric instabilities independent of energy spread in the distribution, while preserving the dynamic aperture. This novel approach promises to reduce particle losses and enable order-of-magnitude increases in beam intensity. In this paper we present results, obtained using the Lie operator formalism, on how chromaticity and dispersion affect particle orbits in integrable optics. We conclude that chromaticity in general breaks the integrability, unless the vertical and horizontal chromaticities are equal. Because of this, the chromaticity correcting magnets can be weaker and fewer correcting magnet families are required, thus minimizing the impact on dynamic aperture.

INTRODUCTION

Nonlinear integrable optics [1] is a concept for mitigating collective instabilities in intense beams using specially designed magnetic elements which introduce large tune spreads with integrable, bounded motion. The lattices which use nonlinear integrable optics are fundamentally different from conventional uses of strong focusing. In conventional strong focusing lattices, nonlinearities introduced by, for example, octupoles for Landau damping are small perturbations on the overall linear dynamics. This is to minimize their impact on the dynamic aperture. For lattices using nonlinear integrable optics, the nonlinear elliptic potential is a dominant part of the dynamics which introduces a large tune spread. This requires particular design considerations to implement properly.

The design of these lattices requires special considerations to ensure that the dynamics remains as close to integrable as possible. In the original treatment of this work, which considers transverse dynamics in the absence of collective effects and zero energy spread, the lattice required a drift section with equal vertical and horizontal beta functions. This was the first design consideration for a nonlinear integrable lattice.

In this proceeding we introduce two additional design criteria based on a study of off-momentum dynamics. Using a Lie map [2] formalism, we derive an expression for the single turn transfer map including the elliptic potential and chromatic and linear dispersive effects. As a result of this calculation, we conclude that chromaticity serves to both modify the invariants and, if not carefully managed, ruin the integrability. Similarly, dispersion through the elliptic element drift breaks the invariant potential. Because the calculations are too elaborate for this proceeding, we here simply state the key results. The detailed work may be found in [3].

BERTRAND-DARBOUX EQUATION

Because the specific conditions of the Bertrand-Darboux equation [4, 5] are critical for understanding the conclusions of this work, we summarize the results here. This is a partial differential equation for a two-dimensional potential $V(x, y)$ which has an invariant of the motion quadratic in the momenta. That is, the Hamiltonian $H = {}^1\!/\!_2(p_x^2 + p_y^2) + V(x, y)$ has a second invariant (aside from the Hamiltonian itself) given by the form

$$\begin{aligned} I = p_x^2 A(x,y) + p_y^2 B(x,y) + p_x p_y C(x,y) + \dots \\ \dots + p_x D(x,y) + p_y E(x,y) + F(x,y). \end{aligned} \quad (1)$$

First, and importantly, we note that H is isotropic in the momenta – the coëfficients of p_x and p_y are equal. This is not generally true; a Hamiltonian for a ring with different vertical and horizontal tunes will have differing coëfficients.

The resulting Bertrand-Darboux differential equation is given by

$$\begin{aligned} xy\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2}\right) + \dots \\ \dots + (y^2 - x^2 + c^2)\frac{\partial^2 V}{\partial x \partial y} + 3y\frac{\partial V}{\partial x} - 3x\frac{\partial V}{\partial y} = 0. \end{aligned} \quad (2)$$

Here we note two more features required of the potential: (1) that the partial differential equation is linear in V and therefore the sum of any set of potentials V_i which satisfy this will have an associated invariant and (2) that only specifically $x^2 + y^2$ satisfies the differential equation, and not x^2 or y^2 individually. Because of (1), a strong focusing lattice with a nonlinear element can form an integrable potential of this form. Because of (2), said strong focusing lattice must have equal vertical and horizontal tunes, a

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restriction already imposed by the assumption of isotropic momenta.

These two requirements will appear as lattice design, where we must tailor certain properties for the lattice to have the desired dynamics. How these are controlled depends on the idea that there are three Tiers of lattice coëxisting, and each one defines certain properties which influence the dynamics of the next Tiers.

THREE TIERS OF LATTICES

In carrying out the calculations for the integrable optics ring, we identify three Tiers of lattice. Each Tier may be thought to interact with the Tiers below it, but not above it. Hence, we may think of Tier 2 as being placed on top of Tier 1, and Tier 3 being placed on top of Tiers 1 and 2. Tier 3 does not affect Tier 1, but Tier 1 directly affects Tier 3. For example, the beta functions are specified by the Tier 1 lattice, which in turn affect the nonlinear integrable elements which make up the Tier 3 lattice.

Tier 1 is the simple linear lattice with linear dispersion. These are the quadrupoles, dipoles, and drifts. This lattice specifies the beta functions, dispersion function, and the tunes that influence our further design. All of the coördinate normalization occurs using the normalizing transformations for the Tier 1 lattice (i.e. the Courant-Snyder parameterization).

Nonlinear effects such as chromaticity and the sextupoles, octupoles, etc. used to adjust the chromaticity are found in Tier 2. These nonlinearities affect the tunes of the Tier 1 lattice, but we assume that they do not affect the choice of normalizing variables. We thus consistently use the normalizing variables derived from the Tier 1 single turn transfer map. Throughout, we include the effects of the nonlinear elements on the chromaticity, but neglect the nonlinearities at higher order in x and y.

This all sets up the relevant parameters for the Tier 3 lattice, which adds the nonlinear elements for the elliptic potential. The parameters for the Tier 1 and Tier 2 lattice must be such that adding the nonlinear elements sets up an integrable potential. Those requirements were summarized above in the discussion of the Bertrand-Darboux equation.

CHROMATICITY & DISPERSION IN THE NONLINEAR INTEGRABLE LATTICE

First of all, our understanding of chromaticity and how to adjust it remains unchanged. Chromatic effects and other nonlinearities for correcting them are found in the Tier 2 lattice. By placing sextupoles π phase advance apart, we can adjust the linear chromaticity while being optically transparent to on-momentum orbits. This pushes the introduction of integrability-breaking nonlinearities to higher order in the transverse coördinates. We can retain this set of tools for nonlinear lattice design. However, the effects of these correction schemes on the dynamic aperture have not been studied in detail, and this will require additional work in the future.

In [3] we detail how the individual lattices can be separated formally using a transfer map approach. This allows us to factor the transfer map cleanly into the Tier 1, Tier 2, and Tier 3 lattices. The combination of these lattices yields the transfer map for the entire lattice and with it the Hamiltonian for the single-turn map. This Hamiltonian is the one described in [1]. It is this Hamiltonian which we must make integrable.

After carrying out the calculations detailed in [3], we obtain the key result that the Hamiltonian for a single turn through the nonlinear integrable ring, including chromaticity and linear dispersion, has for its Hamiltonian

$$\begin{aligned} \mathcal{H} = \frac{\mu_x}{2}\left(1 - C_x(\delta)\right)\left(\overline{p}_x^2 + \overline{x}^2\right) + \dots \\ \dots + \frac{\mu_y}{2}\left(1 - C_y(\delta)\right)\left(\overline{p}_y^2 + \overline{y}^2\right) + \dots \\ \dots + \int_0^\ell ds' \frac{t}{\beta(s')}\mathcal{U}\left(\overline{x} - \frac{\eta(s')\delta}{\sqrt{\beta(s')}}, \overline{y}\right) + \frac{1}{2}\alpha_C\delta^2. \end{aligned} \tag{3}$$

where here μ is the phase advance of the entire ring, C is the dispersion to all orders in δ, and the integral is over the length of the elliptic magnet, with β the equal beta functions through the drift and η the dispersion.

This Hamiltonian clearly violates the assumptions for the Bertrand-Darboux equation. First, the dispersion breaks the analytical form of the elliptic potential. Second, the vertical and horizontal chromaticities violate the isotropic requirement on the momentum and the linear potential. This leads us to two important design considerations to leading order in the Hamiltonian:

1. The vertical and horizontal tunes and chromaticities must be equal.

2. The elliptic element must be located in a dispersion-free drift to maintain the properties of the elliptic potential

If these considerations are satisfied, we get the simplified, integrable Hamiltonian:

$$\begin{aligned} \mathcal{H} = \frac{1}{2}\left(1 - \frac{\mu_0 C(\delta)}{\nu_0}\right)\left(\overline{p}_x^2 + \overline{x}^2 + \overline{p}_y^2 + \overline{y}^2\right) + \dots \\ \dots + t\mathcal{U}(\overline{x}, \overline{y}) + \frac{1}{2}\alpha_C\delta^2. \end{aligned} \tag{4}$$

Here, μ_0 is the phase advance for the entire ring, while ν_0 is the phase advance across the drift where the nonlinear element will be located. For the typical lattice designs, $\mu_0 = 2\pi N + \nu_0$ which allows us to remove the first $2\pi N$ phase advance for the on-momentum oscillations since this is simply the identity.

CONCLUSION

Nonlinear integrable optics is a promising step towards high intensity beams for a variety of research and industrial applications. Because they are fundamentally different from our conventional linear strong focusing lattices, they

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require specific designs to satisfy the integrability conditions. These design conditions, analogous to rules such as not operating a linear lattice near low order rational tunes, must be developed to design a functional integrable lattice.

We have established that chromaticity and its correction schemes are the same as for conventional strong focusing lattices. Our understanding and tools for dealing with these things are unchanged by introducing the elliptic element in the Tier 3 lattice. However, their effects on dynamic aperture remain unknown and should be the subject of future work.

We have presented two new such conditions, related to the off-momentum dynamics. To preserve the $x - y$ isotropy of the linear parts of the lattice, we must have not only equal tunes but equal chromaticities. To keep to the form of the elliptic potential, the nonlinear element must be placed in a drift with no dispersion. These are the first results for the nonlinear integrable optics which consider the longitudinal single-particle dynamics.

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