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CONTROL OF CALCULATIONS IN THE BEAM DYNAMICS USING APPROXIMATE INVARIANTS

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Abstract

One of the important problems in the theory of dynamical systems is to find corresponding (invariants). In this article we are discussing some problems of computing of invariant functions (invariants) for dynamical systems. These invariants can be used for describing of particle beams systems. The suggested method is constructive and it is based on the matrix formalism for Lie algebraic tools. We discuss two types of invariants: kinematic and dynamic. All calculations can be realized in symbolic forms. In particular kinematic invariants are based on the theory of representations of Lie algebras (in particular, using the Casimir's operators). For the case of nonlinear kinematic invariants we propose a recursive scheme, which can be implemented in symbolic forms using instruments of computer algebra (for example, such packages as Maple or Mathematica). The corresponding expressions for invariants can be used to control the correctness of computational experiments, first of all for long time beam dynamics.

INTRODUCTION

As is known, in the theory of dynamical systems, one important task is to find functions $I(\mathbf{X},t)$, which keep the constant value on the trajectories of the system — the so-called invariant functions or simply invariants. In this paper we discuss some issues related to the axiomatic and the computing problems for building of invariants of dynamical systems used in particular to describe the systems controlling beams of particles. The proposed methods are new and constructive. Moreover, they have the form of linear algebraic equations, that allows to easily solve them with the help of computer algebra in two steps. On the first step we solve abstract algebraic equations of corresponding dimension and the results are entered into the appropriate database. On the second step we substitute the parameters of the studying dynamical systems, and then the corresponding dynamical invariants are calculated with a necessary accuracy. The another type of invariants – kinematic invariants are constructed using an algorithm based on the theory of representations of Lie algebras (in particular, using the Casimir operators [1]). Computation of both linear and nonlinear kinematic invariants is performed close to the schemes described in [2,3]. But there we held a clearer description of the computational scheme and its rationale. For the case of nonlinear kinematic invariants proposed scheme is recursive (compare with [2,3]) and also can be easy implemented using computer algebra methods. It should also indicate the need for further study of the problems of constructing nonlinear invariants primarily in terms of differential geometry.

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BASIC CONCEPTS AND DEFINITIONS

We can give the following definition of a dynamical system

Definition 1 *Under the dynamical system with control we mean the mapping*

$$\mathcal{M}:\ \mathfrak{X}\times\mathfrak{U}\times\mathfrak{B}\times\mathfrak{T}\mapsto\mathfrak{X}, \tag{1}$$

where $\mathfrak{U}$, $\mathfrak{B}$, $\mathfrak{T}$ are an admissible control set, a set of control parameters and a set of finite measure from R^1 respectively.

So, let us defined a semigroup of symmetry $\mathfrak{D}=\{\mathcal{D}\}$ for this dynamic system, i. e. our dynamical system with control $\mathbf{U}$ is given by the equation of motion

$$\mathbf{X}=\mathcal{F}\left(\mathbf{X},\mathbf{U},\mathbf{B},t\right), \tag{2}$$

and in new variables (after conversion of symmetry) $\mathcal{D}=\mathcal{A}_{\mathfrak{X}}\otimes\mathcal{A}_{\mathfrak{U}}\otimes\mathcal{A}_{\mathfrak{B}}\otimes\mathcal{A}_{\mathfrak{T}})$ we will obtain

$$\mathbf{Y}=\mathcal{Y}\left(\mathcal{Y},\mathbf{U},\mathbf{B},t\right).$$

The map $\mathcal{Y}$ corresponds, in particular, the procedure of observation (measurement) of state of the dynamic system. In particularly the investigated system can be described by the system of ordinary differential equation

$$\frac{d\mathbf{X}}{dt}=\mathbf{F}(\mathbf{X},\mathbf{U},\mathbf{B},t), \tag{3}$$

or we can introduce the following integral operator

$$\mathcal{F}(\mathbf{X},\mathbf{U},\mathbf{B},t)=\mathbf{X}_0+\int\limits_{t_0}^{t}\mathbf{F}(\mathbf{X}(\tau),\mathbf{U}(\tau),\mathbf{B},\tau)d\tau. \tag{4}$$

We can give the following definition

Definition 2 *Symmetry transformation of a dynamical system with control will be called a set of maps $\mathcal{A}_{\mathfrak{T}}:\mathfrak{T}\to\tilde{\mathfrak{T}}$, $\mathcal{A}_{\mathfrak{X}}:\mathfrak{X}\to\tilde{\mathfrak{X}}$, $\mathcal{A}_{\mathfrak{U}}:\mathfrak{U}\to\tilde{\mathfrak{U}}$, $\mathcal{A}_{\mathfrak{B}}:\mathfrak{B}\to\tilde{\mathfrak{B}}$, $\mathcal{A}_{\mathfrak{Y}}:\mathfrak{Y}\to\tilde{\mathfrak{Y}}$, providing the commutativity of the following diagrams:*

$$\begin{array}{ccccccc}\mathfrak{X}\times\mathfrak{U}\times\mathfrak{B}\longrightarrow\mathfrak{X}, & & \mathfrak{X}\times\mathfrak{U}\times\mathfrak{B}\longrightarrow\mathfrak{Y},\\ \downarrow\ \downarrow\ \downarrow\quad\ \downarrow & & \downarrow\ \downarrow\ \downarrow\quad\ \downarrow\\ \tilde{\mathfrak{X}}\times\tilde{\mathfrak{U}}\times\tilde{\mathfrak{B}}\longrightarrow\tilde{\mathfrak{X}}, & & \tilde{\mathfrak{X}}\times\tilde{\mathfrak{U}}\times\tilde{\mathfrak{B}}\longrightarrow\tilde{\mathfrak{Y}}.\end{array}$$

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The definition (including structural arrangement) transformations leaving invariant system of equations (3), is a very important task, which allows to solve many of the problems in the theory of dynamical systems. In particular, we recall that this approach can greatly simplify the task of integrating the equations of motion [4].

It is worth mentioning that the semigroups and groups of symmetries can be both continuous and discrete. Theory of continuous groups (in particular, Lie groups) is well known and widely used in various fields of natural science [5]. Among of continuous transformations particularly noteworthy group of symplectic transformations, which preserve the Poisson structure of dynamical systems. It is known, in particular, that the Hamiltonian system itself generates symplectic transformation. Returning to the problem of invariants, we recall that an invariant dynamical system is any function (sufficiently smooth), if the following condition takes place

$$\frac{d}{dt}\Big(I(\mathbf{X},t)\Big)\Big|_{\mathbf{X}=\mathcal{F}(\mathbf{X},\mathbf{U},\mathbf{B},t)} = 0. \tag{5}$$

In the paper [2] there are introduced the concepts of kinematic and dynamic invariants (for example for symplectic maps). We generalize these definitions for arbitrary dynamical systems, as well as arbitrary transformations (not only generated by the dynamical system) and to demonstrate how to use this methods for investigation of particle beam problems.

Recall that the dynamic system, and whether symmetry (infinitesimal) is generated by the vector fields – operators. Let a dynamic system describes by the following operator Lie $\mathcal{L}_F$:

$$\mathcal{L}_\mathbf{F} = \mathbf{F}^*(\mathbf{X},t)\frac{\partial}{\partial\mathbf{X}} = \sum_{k=0}^{\infty}\mathbf{F}_k^*(\mathbf{X},t)\frac{\partial}{\partial\mathbf{X}} = \\ = \sum_{k=0}^{\infty}\left(\mathbf{X}^{[k]}\right)^*\mathbb{F}_k^*(t)\frac{\partial}{\partial\mathbf{X}} \tag{6}$$

$$\mathcal{L}_\mathbf{G} = \mathbf{G}^*(\mathbf{X},t)\frac{\partial}{\partial\mathbf{X}} = \sum_{k=0}^{\infty}\mathbf{G}_k^*(\mathbf{X},t)\frac{\partial}{\partial\mathbf{X}} = \\ = \sum_{k=0}^{\infty}\left(\mathbf{X}^{[k]}\right)^*\mathbb{G}_k^*(t)\frac{\partial}{\partial\mathbf{X}}, \tag{7}$$

and for the symmetry (local) group – the Lie operator $\mathcal{L}_\mathbf{G}$: where $\mathbf{F}_k, \mathbf{G}_k$ are homogeneous vectors of degree k-th order polynomials in the phase variables. From the General theory of groups and algebras follows that the lie group generated Lie algebra of operators $\mathcal{L}_\mathbf{G}$, were a group of symmetry, it is necessary and sufficient to satisfy the following equality

$$\{\mathcal{L}_\mathbf{F}, \mathcal{L}_\mathbf{G}\} = 0,$$

whence it follows

$$[\mathbf{G},\mathbf{F}] = \{\mathcal{L}_\mathbf{G}, \mathcal{L}_\mathbf{F}\} = \mathcal{L}_\mathbf{G}\circ\mathbf{F} - \mathcal{L}_\mathbf{F}\circ\mathbf{G} = 0. \tag{8}$$

The eq. (8) is called the determining equation for the group symmetry generated by the function $\mathbf{G}$, and represents a system of linear inhomogeneous equations in partial derivatives of the first order for component vector-functions $\mathbf{G}(\mathbf{X},t) = \{g_j(\mathbf{X},t)\}_{j=\overline{1,n}}$. Decompositions of (6) and (7) after substitution into (8) lead us to the equation

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\mathbf{G}_k^*\frac{\partial}{\partial\mathbf{X}}\mathbf{F}_j - \mathbf{F}_j^*\frac{\partial}{\partial\mathbf{X}}\mathbf{G}_k\right) = 0,$$

where, given the homogeneity of the polynomials $\mathbf{G}_k, \mathbf{F}_j$, we obtain the following system of constitutive equations:

$$\sum_{j=0}^{k}\left(\mathbf{F}_j^*\frac{\partial}{\partial\mathbf{X}}\mathbf{G}_{k-j} - \mathbf{G}_{k-j}^*\frac{\partial}{\partial\mathbf{X}}\mathbf{F}_j\right) = \\ = \sum_{j=0}^{k}\left[\mathbf{F}_j,\mathbf{G}_{k-j}\right] = 0, \ \forall\ k \geq 0. \tag{9}$$

We note that for $k = 0$ the equality (9) is performed automatically: $\partial\mathbf{G}_0/\partial\mathbf{X} = \partial\mathbf{F}_0/\partial\mathbf{X} = 0$. For $k = 1$ one obtains

$$\mathbf{F}_0^*\frac{\partial}{\partial\mathbf{X}}\mathbf{G}_1 = -\mathbf{G}_0^*\frac{\partial}{\partial\mathbf{X}}\mathbf{F}_1. \tag{10}$$

It is easy to see that in the general case we have (for $j \geq 1$)

$$\mathbf{F}_0^*\frac{\partial}{\partial\mathbf{X}}\mathbf{G}_j = \left[\mathbf{F}_0,\mathbf{G}_j\right] = -\sum_{k=1}^{j}\left[\mathbf{F}_k,\mathbf{G}_{j-k}\right]. \tag{11}$$

The right part of eq. (11) is calculated by recurrent way, beginning from the vector $\mathbf{G}_0(t)$ (see (10)). So, solving the equation (11) with respect to $\mathbf{G}_j$, we can construct the vector-functions $\mathbf{G}_j$, $j \geq 1$, are included in the definition infinitesimal operator of the symmetry group of $\mathcal{L}_\mathbf{G}$.

Naturally, from a practical point of view, we must confine ourselves up to some order in the expansions (6) and (7), which will automatically lead to the fact that equation (8) is satisfied approximately. We can give in accordance with this the definition of approximate symmetry (and respectively approximating the infinitesimal operator of the symmetry group).

Definition 3 *Let $\mathcal{L}_\mathbf{F}$, $\mathcal{L}_\mathbf{G}$ — dynamic system operators and symmetry groups, respectively, and we have an equality $\{\mathcal{L}_\mathbf{F}, \mathcal{L}_\mathbf{G}\} = \mathcal{L}_\mathbf{H}$, where $\mathbf{H} = \sum_{k=0}^{\infty}\mathbf{H}_k = \sum_{k=0}^{\infty}\mathbb{H}_k\mathbf{X}^{[k]}$. Then, if $\mathbf{H}_k \equiv 0$ (or $\mathbb{H}_k \equiv 0$) for all $k \leq N$, then we say that $\mathcal{L}_\mathbf{G}$ generates approximate symmetry up to order N.*

One can prove the following lemma [6]

Lemma 1 *The set of approximate N-th order symmetries for the vector field $\mathcal{L}_\mathbf{F} = \mathbf{F}^*\partial/\partial\mathbf{X}$ forms a Lie algebra $\mathfrak{LS}_\mathbf{F}^{(N)}$.*

Lemma 2 *For Lie algebras of approximate symmetries there is an embedding*

$$\mathfrak{LS}_\mathbf{F}^{(N+1)} \subseteq \mathfrak{LS}_\mathbf{F}^{(N)}.$$

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We introduce also the following definitions

Definition 4 *Function* $\mathbf{I}^{\mathbf{F}}$ *is called an invariant dynamical system, if the equation* $\mathcal{L}_{\mathbf{F}} \circ \mathbf{I}^{\mathbf{F}} = 0$ *is fulfillment.*

and

Definition 5 *Function* $\mathbf{I}^{\mathbf{F}^{(N)}}$ *is called an approximate* N*-th order invariant dynamical system if there is the equality* $\mathcal{L}_{\mathbf{F}} \circ \mathbf{I}^{\mathbf{F}^{(N)}} = \sum_{k=N+1}^{\infty} \mathbf{H}_k$.

SOME COMPUTATIONAL EXAMPLES

Using the properties of Lie operators and Kronecker operations, it is easy to get

$$\left[\mathbb{F}_0, \mathbb{G}_j \mathbf{X}^{[j]}\right] = \mathcal{L}_{\mathbf{F}_0} \circ \mathbb{G}_j \mathbf{X}^{[j]} = \mathbb{G}_j \mathbb{F}_0^{\oplus j} \mathbf{X}^{[j-1]},$$

$$\begin{aligned} &\left[\mathbb{F}_k \mathbf{X}^{[k]}, \mathbb{G}_{j-k} \mathbf{X}^{[j-k]}\right] = \\ &= \mathcal{L}_{\mathbf{F}_k} \circ \mathbb{G}_{j-k} \mathbf{X}^{[j-k]} - \mathcal{L}_{\mathbf{G}_{j-k}} \circ \mathbb{F}_k \mathbf{X}^{[k]} = \\ &= \left(\mathbb{G}_{j-k} \mathbb{F}_k^{\oplus(j-k)} - \mathbb{F}_k \mathbb{G}_{j-k}^{\oplus k}\right) \mathbf{X}^{[j-1]}, \quad 1 \le k \le j, \end{aligned}$$

whence it follows that the matrix equality (for $1 \le k \le j$)

$$\mathbb{G}_j \mathbb{F}_0^{\oplus j} = -\sum_{k=1}^{j} \left(\mathbb{G}_{j-k} \mathbb{F}_k^{\oplus(j-k)} - \mathbb{F}_k \mathbb{G}_{j-k}^{\oplus k}\right). \tag{12}$$

Let us consider the (12) more precisely. Let $j = 1$, then from (12) we obtain

$$\mathbb{G}_1 \mathbb{F}_0 = \mathbb{F}_1 \mathbb{G}_0 - \mathbb{G}_0 = \mathbb{G}_0 \left(\mathbb{F}_1 - \mathbb{E}\right). \tag{13}$$

For $j = 2$ we obtain

$$\left(\mathbb{F}_2 \mathbb{G}_0^{\oplus 2} - \mathbb{G}_0 \mathbb{E}\right) = 0. \tag{14}$$

We should note, that in many practical problems we have $\mathbb{F}_0 = \mathbf{F}_0 = 0$ and $\det(\mathbb{F}_1 - \mathbb{E}) \ne 0$. In this case we can simplify the obtained equations: $\mathbb{G}_0 = \mathbb{F}_0 = 0$ and $(\mathbb{F}_1 \mathbb{G}_1 - \mathbb{G}_1 \mathbb{F}_1) = 0$.

Note 1. It is easy to see that the eq. (14) can be allowed by the substitution $\mathbb{G}_1 = \alpha \mathbb{E} + \beta \mathbb{F}_1$, where α and β are arbitrary constants. Indeed, in this case, eq. (14) becomes identical.

For $j = 3$ we obtain $\mathbb{F}_1 \mathbb{G}_2 - \mathbb{G}_2 \mathbb{F}_1^{\oplus 2} + \mathbb{F}_2 \mathbb{G}_1^{\oplus 2} - \mathbb{G}_1 \mathbb{F}_2 = 0$. It should note that in this case we obtain the well known matrix equations $\mathbb{A}\mathbb{X} + \mathbb{X}\mathbb{B} = \mathbb{C}$, where $\mathbb{X} = \mathbb{G}_2$, $\mathbb{A} = \mathbb{F}_1$, $\mathbb{B} = -\mathbb{F}_1^{\oplus 2} = -\mathbb{A}^{\oplus 2}$, $\mathbb{C} = \mathbb{G}_1 \mathbb{F}_2 - \mathbb{F}_2 \mathbb{G}_1^{\oplus 2}$ or

$$\left(\mathbb{E}^{[2]} \otimes \mathbb{F}_1 - \left(\mathbb{F}_1^{\oplus 2}\right)^* \otimes \mathbb{E}\right) \text{vect}\, \mathbb{X} = \text{vect}\, \mathbb{C}, \tag{15}$$

where $\mathbb{E}$ is the identity matrix of the necessary dimension and $\otimes$ is the Kronecker multiplication. For unique solution of the eq. (15) for any matrices $\mathbb{C}$, it is necessary and sufficient to satisfy the following inequality

$$\lambda_i - \mu_k \ne 0 \;\; \forall\, i, k, \tag{16}$$

where λ_i, and μ_k — eigenvalues of $\mathbb{F}_1$ and $\mathbb{F}_1^{\oplus 2}$ correspondingly. However, for μ_k we can write $\{\mu_k\} = \{\lambda_i + \lambda_j\}$, i. e is eigenvalues values of $\mathbb{F}_1^{\oplus 2}$ consists of all pairwise sums of eigenvalues of the matrix $\mathbb{F}_1$. Thus, the condition (16) can be rewritten in the following form:

$$\lambda_i - \left(\lambda_j + \lambda_k\right) \ne 0 \;\; \forall\, i, j, k, \tag{17}$$

where $\{\lambda_i\}$ — a set of of eigenvalues of the matrix $\mathbb{F}_1$. These equations can be solved for some control elements (for example, for quadrupole lenses). Obviously, that the condition (17) holds, so that equation (15) is solvable

$$\begin{aligned} \text{vect}\, \mathbb{G}_2 = \left(\mathbb{E}^{[2]} \otimes \mathbb{F}_1 - \left(\mathbb{F}_1^{\oplus 2}\right)^* \otimes \mathbb{E}\right)^{-1} \times \\ \times \text{vect}\, \left(\mathbb{G}_1 \mathbb{F}_2 - \mathbb{F}_2 \mathbb{G}_1^{\oplus 2}\right). \end{aligned}$$

Note 2. If it is required to carry out some additional conditions, then we obtain equations for the coefficients α and β. For example, for symplectic properties, coefficients α, β can be calculated from the equation

$$\left(\alpha^2 - 1\right) \mathbb{J}_0 + \alpha\beta \left(\mathbb{F}_1^* \mathbb{J}_0 + \mathbb{J}_0 \mathbb{F}_1\right) + \beta^2 \mathbb{F}_1^* \mathbb{J}_0 \mathbb{F}_1 = 0.$$

For example, for a quadrupole lens with gradient k one can obtain the following equality for coefficients α and β.

$$\alpha^2 + \beta^2 k = 0.$$

CONCLUSION

The above described approach allows us to constructive computational procedures for computation of approximate invariants (for investigated beam lines) up to necessary order of nonlinearities.

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