THEORY AND SIMULATION OF EMITTANCE GROWTH CAUSED BY SPACE CHARGE AND LATTICE INDUCED RESONANCES

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Abstract

Emittance growth and beam loss in high intensity circular proton accelerators are one of the most serious issue which limit their performance. The emittance growth is caused by linear and nonlinear resonances of betatron/synchrotron oscillation due to lattice and space charge nonlinear force. We should first understand which resonances how are serious. Tuning shift and resonance strength induced by space charge and lattice nonlinearity is discussed with integrals along a ring like the radiation integrals. Emittance growth is discussed using so-called standard model, in which the resonance width has an important role.

INTRODUCTION

Particles move with experience of electro-magnetic field of lattice elements and space charge. We study slow emittance growth arising in a high intensity circular proton ring. We assume that the beam distribution is static, and each particle moves in the filed of the static distribution. As a practical issue, we concern about beam loss of 0.1-1% during a long term (≈ 10,000 turns) in J-PARC MR. A halo is formed by the nonlinear force due to the electro-magnetic field. The halo, which consists of small part of whole beam, does not affect the field. Particle motion is described by a single particle Hamiltonian in the field.

Hamiltonian is separated by three parts for (1) linear betatron motion (μJ) (2) nonlinear component of the lattice magnets (U_{nl}) and (3) space charge potential (U_{sc}).

\[ H = \mu J + U_{nl} + U_{sc}. \] (1)

where Hamiltonian is represented by action variables \( J \) and \( \phi \), which are Courant-Snyder invariant (W = 2J) and betatron phase, respectively. Betatron motion is expressed by the action variables as,

\[ x(s)_\beta = \sqrt{2\beta_x(s) J_x \cos(\phi_x(s))}, \]
\[ y(s)_\beta = \sqrt{2\beta_y(s) J_y \cos(\phi_y(s))}. \] (2)

where dispersion is subtracted in \( x \) and \( y \). Betatron phase advance per turn (\( \mu_i = 2\pi \tilde{\gamma}_i, i = x, y \)), which depends on \( J_i \) due to \( U \)'s, is given by

\[ \mu_i(J) = \phi_i(s + L) - \phi_i(s) = \frac{\partial H}{\partial J_i} = \mu_i + \frac{\partial (U_{nl} + U_{sc})}{\partial J_i}. \] (3)

where \( J = (J_x, J_y) \). Synchrotron motion is treated as external given motion for \( s \),

\[ z = z_0 \cos(2\pi \nu_s s/L). \] (4)

Hamiltonian is expanded by Fourier series [1],

\[ H = \mu J + U_{00}(J) + \sum_{m_x, m_y} U_{m_x, m_y}(J) \exp(-im_x \phi_x - im_y \phi_y) \] (5)

First and second terms in RHS characterize shift, spread and slope of tune.

\[ \mu_i = \frac{\partial H}{\partial J_i} = \mu_i + \frac{\partial U_{00}}{\partial J_i} \] (6)

Third term is averaged out for the tune shift due to the betatron phase variation. Resonance condition is expressed by

\[ m_x \mu_x(J) + m_y \mu_y(J) = 2\pi n. \] (7)

where \( n \) is an integer. The resonance condition Eq.(7) gives a line in \((J_x, J_y)\) space. \( J \) satisfying Eq.(7) is expressed by \( J_R \).

Hamiltonian is expanded near \( J_R \) as

\[ U_{00}(J) = U_{00}(J_R) + \frac{\partial U_{00}}{\partial J} \bigg|_{J_R} (J - J_R) \]
\[ + (J - J_R)^2 \frac{\partial^2 U_{00}}{\partial J^2} \bigg|_{J_R} (J - J_R) \] (8)

Third term in RHS is characterized by the tune slope

\[ \frac{\partial \nu_x}{\partial J_x} = \frac{\partial \nu_y}{\partial J_y} = \frac{\partial^2 U_{00}}{\partial J_x \partial J_y} \] (9)

Canonical transformation for new variable \( P \) and \( \psi \) is considered

\[ F_2(P, \psi) = (J_x R + m_x P_1 + m_{x,2} P_2) \phi_x \]
\[ + (J_y R + m_y P_1 + m_{y,2} P_2) \phi_y \] (10)

Choose of \((m_{x,2}, m_{y,2})\) is arbitrary under independent of \((m_x, m_y)\). Choosing \( m_{x,2} = 0, m_{y,2} = 1 \),

\[ P_1 = \frac{J_x R}{m_x} \psi_1 = m_x \phi_x + m_y \phi_y \] (11)

\[ P_2 = \frac{(J_y R)}{m_x} (J_x R - J_x R) \psi_2 = \phi_y \]

Hamiltonian for \( J \) dependent terms is now given by

\[ H_{00} = U_{00} \approx \frac{\Lambda}{2} P_1^2, \] (12)

where

\[ \Lambda \equiv m_x^2 \frac{\partial^2 U_{00}}{\partial J_x^2} + m_{x,2} m_y \frac{\partial^2 U_{00}}{\partial J_x \partial J_y} + m_{y,2} \frac{\partial^2 U_{00}}{\partial J_y^2}. \] (13)

ISBN 978-3-95450-136-6
The resonance term, which is third term of RHS in Eq.(5), drives resonances. The resonance strengths $U_m$ as function of $J$ are approximated to be those at $J_R$

$$U_m(J) \approx U_m(J_R) \quad m = (m_x,m_y).$$

(14)

Hamiltonian for the standard model is given as

$$H = \frac{\Lambda}{2} P_1^2 + U_m(J_R) \cos \phi_1.$$  (15)

Phase space structure near resonances are characterized by the resonance width. The resonance width is given by

$$\Delta p_1 = 4 \sqrt{\frac{U_m}{\Lambda}} \quad \Delta J_x = 4 m_x \sqrt{\frac{U_m}{\Lambda}}.$$  (16)

**EVALUATION OF RESONANCE WIDTH**

**Resonances Due to Space Charge Force**

We first discuss the space charge potential $U_{sc}$ [2]. Beam distribution is assumed to be Gaussian in transverse determined by emittance and Twiss parameters. $U$ contains linear component, which gives a tune shift and Twiss parameter distortion.

$$U_{sc}(s) = \int ds' U_{sc}(s',s) = -\frac{\lambda_p r_p}{\beta^2 y^3} \int ds'$$

$$\int_0^\infty 1 - \exp \left(-\frac{x(s',s)^2}{2\sigma_x^2} + \frac{y(s',s)^2}{2\sigma_y^2} \right) du \right)$$

$$\sqrt{2\sigma_x^2 + u} \sqrt{2\sigma_y^2 + u}$$

(17)

$x$ and $y$ are sum of betatron coordinates dispersion orbit at $s'$ as

$$x(s',s) = \sqrt{2} \beta_x s' \cos(\phi_x(s',s) + \phi(x(s',s))) + \eta(s') \delta(s)$$

$$y(s',s) = \sqrt{2} \beta_y s' \cos(\phi_y(s',s) + \phi_y(s)).$$

(18)

where $\phi_x,y(s',s)$ is the betatron phase difference between $s$ and $s'$ and $\eta$ is the dispersion. $\delta$ is a given function of $s$,

$$\delta(s) = \delta_0 \sin \mu_s s/L.$$  (19)

The synchrotron motion is slow $v_x \approx 0.002$ for J-PARC MR. We consider $\delta$ changes additivitically in the evaluation of the tune shift and resonance terms: i.e. they are evaluated at a given $\delta$. We introduce folllowing variables.

$$\frac{x^2}{2\sigma_x^2 + u} = w_{x\eta} + w_x \cos 2(\phi_x + \phi_x) + v_x \cos(\phi_x + \phi_x)$$

$$\frac{y^2}{2\sigma_y^2 + u} = w_y + w_y \cos 2(\phi_y + \phi_y).$$

(20)

where

$$w_x = \frac{\beta_x J_x}{2\sigma_x^2 + u}, \quad w_{x\eta} = \frac{\beta_x J_x + \frac{\eta^2}{2} \delta^2}{2\sigma_x^2 + u}.$$  (21)

Using

$$e^{-w \cos \phi} = \sum_{k=-\infty}^{\infty} (-1)^k I_k(w) e^{-ik\phi},$$

(23)

the exponential term is expressed by modified Bessel functions,

$$\exp\left(\frac{x^2}{2\sigma_x^2 + u} - \frac{y^2}{2\sigma_y^2 + u}\right) = e^{-w_{x\eta} - w_y}$$

$$\sum_{k,l=-\infty}^{\infty} (-1)^{l+k} I_l(w_x) I_l(w_y) e^{-i2(l+k)(\phi_x + \phi_y)}.$$  (24)

The Fourier component, which correspond to resonance strength, is given by

$$U_{m_x,m_y}(J_x,J_y) = -\frac{\lambda_p r_p}{\beta^2 y^3} \int ds \int_0^\infty \frac{dt}{\sqrt{2\sigma_x^2 + u} \sqrt{2\sigma_y^2 + u}}$$

$$\left[ \delta_{m_x,0} \delta_{m_y,0} - \exp(-w_{x\eta} - w_y) \sum_{l=-\infty}^{\infty} (-1)^{(m_x+l)(m_y+l)/2} I_l(w_x) I_l(w_y) e^{-im_x\phi_x - im_y\phi_y} \right].$$  (25)

where $m_x + l$ and $m_y$ are even numbers.

The tune slope $\partial U_{00}/\partial J_x^2$ induced by space charge potential is evaluated by $U_{00}(J_x,J_y)$ in Eq.(27).

$$U_{00}(J_x,J_y) = -\frac{\lambda_p r_p}{\beta^2 y^3} \int ds \int_0^\infty \frac{dt}{\sqrt{2\sigma_x^2 + u} \sqrt{2\sigma_y^2 + u}}$$

$$\left[ 1 - e^{-w_{x\eta} - w_y} \sum_{l=-\infty}^{\infty} (-1)^l I_l(w_x) I_l(w_y) e^{-im_x\phi_x - im_y\phi_y} \right].$$  (26)

where $t = u/\sigma_x^2$ and $r_{xy} = \sigma_y^2/\sigma_x^2$ and

$$\frac{\partial}{\partial J_x} \frac{\partial}{\partial J_y} = \frac{\beta_x}{2\sigma_x^2} \frac{\partial}{\partial w_x} \frac{\partial}{\partial w_y}.$$  (27)

The tune shift for on energy particle ($\delta = 0$) is given by derivative of $U_{00}$ for $J_{xy}$ as follows,

$$2\pi \Delta v_x = \frac{\partial U_{00}}{\partial J_x}$$

$$= -\frac{\lambda_p r_p}{\beta^2 y^3} \int ds \frac{\partial}{\partial x} \frac{\sigma_x^2}{2\sigma_x^2 + u} \int_0^\infty \frac{e^{-w_{x\eta} - w_y} dt}{(2 + t)^{3/2}(2r_{xy} + t)^{1/2}}$$

$$\left[ (I_0(w_x) - I_1(w_x))I_0(w_y) \right],$$  (28)

$$2\pi \Delta v_y = \frac{\partial U_{00}}{\partial J_y}$$

$$= -\frac{\lambda_p r_p}{\beta^2 y^3} \int ds \frac{\partial}{\partial x} \frac{\sigma_x^2}{2\sigma_x^2 + u} \int_0^\infty \frac{e^{-w_{x\eta} - w_y} dt}{(2 + t)^{3/2}(2r_{xy} + t)^{1/2}}$$

$$\left[ I_0(w_x)(I_0(w_y) - I_1(w_y)) \right],$$  (29)
The tune slope for $\delta = 0$ is given by second derivative of $U_{00}$ as follows,

$$\frac{\partial^2 U_{00}}{\partial J_x^2} = 2\pi \frac{\partial \nu_x}{\partial J_x}$$

$$= \frac{\lambda_p r_p}{\beta_x^3} \int ds \frac{\beta_x^2}{\sigma_x^4} \int_0^\infty \frac{e^{-w_x-w_y}}{(2 + t)^{\frac{3}{2}}(2r_{yx} + t)^{\frac{3}{2}}} \left\{ \frac{3}{2} I_0(w_x) - 2 I_1(w_x) + \frac{1}{2} I_2(w_x) \right\} I_0(w_y) ,$$

where formulae $I_0(x)' = I_1(x)$, $I_0(x)'' = (I_0(x) + I_2(x))/2$ are used.

The integral is performed using linear lattice information of J-PARC MR. Figure 1 shows tune spread ($\Delta \nu_{x,y} (J_x, J_y)$), slope ($\partial^2 U_{00}/\partial J_x^2$).

Figure 2 shows the tune slope ($U_{ij} = \partial^2 U_{00}/\partial J_i \partial J_j$).

Figures 3 and 4 show the resonance strengths of $U_{30}$ and $U_{40}$ for $(\nu_x, \nu_y) = (21.39, 21.43)$ and $(22.40, 20.75)$, respectively. For $U_{30}$, $l = 1$ term is taken with an energy deviation $\delta = \sigma_\delta = 0.1\%$. The resonance strength is an order of $10^{-7}$ m.

**Resonance Strength Taking Into Account of Superperiodicity**

J-PARC MR ring has superperiodicity of three, therefore resonances without $m_x \nu_x + m_y \nu_y = 3n$ is suppressed. It is sufficient to consider 1/3 ring under the perfect superperiodicity. New operating point, $(\nu_x, \nu_y) = (21.39, 21.43)$, $(\nu_x/3, \nu_y/3) = (7.13, 7.143)$. While present operating point, $(\nu_x, \nu_y) = (22.40, 20.75)$, $(\nu_x/3, \nu_y/3) = (7.4667, 6.9167)$, Higher order resonances are concerned now.

Figure 5 and 6 shows resonance strengths for $(\nu_x/3, \nu_y/3) = (7.13, 7.143)$ and $(\nu_x/3, \nu_y/3) = (7.4667, 6.9167)$, respectively. The resonance strengths are order of $10^{-9}$ m, while they are $10^{-7}$ m in above case of Figs.3 and 4. Since the resonance width scales square root of the strength, the width is one order smaller than that of above case.
Resonances Due to Nonlinear Magnets

Tune spread/slope and resonances are also induced by nonlinear magnets. One turn map is expanded by 12-th order polynomials for J-PARC MR. Taking at phase independent term, \( U_{00} \) is obtained as

\[
U_{00}(J) = 3.43100 \times 10^4 L_c^2 + 7.09314 \times 10^4 L_c J_x + 1.75729 \times 10^4 J_x^2 + 2.84124 \times 10^3 J_x^2 J_y + \\
7.07331 \times 10^3 J_x^2 J_y + 6.98232 J_x^2 J_y + 7.58773 \times 10^3 J_x^2 J_y + \\
5.7318 \times 10^3 J_x^2 J_y + 4.35629 \times 10^3 J_x J_y^2 + 6.0595 J_x J_y^2 + 2096.0 J_x J_y + \\
4.1129 \times 10^3 J_x J_y^2 + 1.09295 \times 10^3 J_x J_y^2 + 5.3027 \times 10^2 J_x J_y^2 + 7.0842 J_x J_y^2 + 1104.3 J_x J_y^2.
\]

(33)

Figure 7 shows the tune shift and slope. Typical tune slope is \( \partial^2 U_{00} / \partial x^2 = 1000 \sim 3000 \). Tune slope of space charge (\( \sim 10,000 \)) is dominant for that of lattice nonlinearity at \( J < 9\sigma(3\sigma) \), while it is similar for \( U_{x_c,00} \) at \( J_x = 100 \mu \text{m} \).

Figure 8: Tune spread (\( \Delta v_{x,y}(J_x, J_y) \)) induced by lattice nonlinearity.

Resonance strength due to lattice nonlinearity is also obtained by the one turn map. Table 1 shows the resonance strength \( U_{m,n}(J) \). up to 4-th. Superperiodicity is not taken into account. The resonance strength, which is order of \( 10^{-7} \text{ m} \), is similar as the space charge induced resonances.

SIMULATION USING THE RESONANCE HAMILTONIAN

Without Synchrotron Motion

We study emittance growth for an accelerator model with a given tune slope and resonance strength. Hamiltonian is written explicitly by real and imaginary parts of \( U_m \).

\[
H = \mu_x J_x + \mu_y J_y + U_0(J_x, J_y) + U_{m,c}(J_x, J_y) \cos m \cdot \phi + U_{m,s}(J_x, J_y) \sin m \cdot \phi.
\]

(34)
where $m\phi = m_x\phi_x + m_y\phi_y$. Symplectic transformation for above Hamiltonian is expressed by

$$\tilde{\phi}_i = \phi_i + \frac{\partial U_0}{\partial J_i} + \frac{\partial U_{m,c}}{\partial J_i} \sin m\phi - \frac{\partial U_{m,s}}{\partial J_i} \cos m\phi$$

$$J_i = J_i - m_i(U_{m,c} \sin m\phi - U_{m,s} \cos m\phi). \quad (35)$$

where $\tilde{J}$ and $\tilde{\phi}$ are those after the transformation, and $U_{m,c}'s$ are function of $J_i$ and $\phi_i$. Second equation of Eq.(35) is implicit relation. To solve $(\tilde{J}, \tilde{\phi})$, Newton-Raphson method is used.

$$f_x = \tilde{J}_x - J_x - m_x(U_{m,c} \sin m\phi - U_{m,s} \cos m\phi) = 0$$

$$f_y = \tilde{J}_y - J_y - m_y(U_{m,c} \sin m\phi - U_{m,s} \cos m\phi) = 0. \quad (36)$$

Iteration of Newton method is expressed by

$$\begin{pmatrix} \tilde{J}_x \\ \tilde{J}_y \end{pmatrix}_{n+1} = \begin{pmatrix} \tilde{J}_x \\ \tilde{J}_y \end{pmatrix}_n - F^{-1} \begin{pmatrix} f_x \\ f_y \end{pmatrix}_n. \quad (37)$$

where $F$ is Jacobian matrix for $f_i$; $F_{ij} = \partial f_i/\partial J_j$, $i, j = x, y$.

The resonance width is estimated by the tune slope and resonance strength seen in Figs.2, 3 and 4. The estimated resonance width agrees well with the width seen in Figure 9.

$$\Delta J_x = 4 \sqrt{\frac{U_m}{\partial^2 U_{00}/\partial J_x^2}} = 4 \sqrt{\frac{10^{-7}}{10^4}} = 12 \times 10^{-6} m = 12 \mu m \quad (38)$$

With Synchrotron Motion

Synchrotron motion is treated as an external force oscillation

$$z = z_0 \cos(\mu_x s/L) = z_0 \cos(\mu_x n_{\text{turn}}) \quad (39)$$

Table 1: $U_{m_x, m_y}(J)$ for Lattice Nonlinearity. $U'$s are evaluated at $J$ 3rd and 4-th column. The suffix, B0,B and BR means lattices without errors, lattice with measured beta and measured beta and coupling [3].

| mx   | my   | Jx   | Jy   | $|U_m|$ (B0) | $|U_m|$ (B) | $|U_m|$ (BR) |
|------|------|------|------|-------------|-------------|-------------|
| 1    | 0    | 3.6E-05 | 0.0E+00 | 4.84E-08 | 1.88E-07 | 1.86E-07 |
| 2    | 0    | 3.6E-05 | 0.0E+00 | 2.47E-08 | 4.55E-08 | 4.66E-08 |
| 1    | 1    | 1.8E-05 | 1.8E-05 | 1.28E-25 | 1.67E-26 | 4.01E-09 |
| 0    | 2    | 0.0E+00 | 3.6E-05 | 5.55E-09 | 3.91E-09 | 2.69E-09 |
| 3    | 0    | 3.6E-05 | 0.0E+00 | 5.46E-08 | 1.29E-07 | 1.32E-07 |
| 2    | 1    | 1.8E-05 | 1.8E-05 | 2.09E-25 | 1.42E-26 | 1.42E-07 |
| 2    | -1   | 1.8E-05 | 1.8E-05 | 2.16E-25 | 4.52E-27 | 7.96E-08 |
| 1    | 2    | 1.8E-05 | 1.8E-05 | 4.66E-08 | 1.78E-07 | 1.83E-07 |
| 1    | -2   | 1.8E-05 | 1.8E-05 | 1.48E-07 | 2.72E-07 | 2.72E-07 |
| 0    | 3    | 0.0E+00 | 3.6E-05 | 1.42E-25 | 1.58E-26 | 1.10E-07 |
| 4    | 0    | 3.6E-05 | 0.0E+00 | 2.50E-07 | 2.51E-07 | 2.51E-07 |
| 3    | 1    | 1.8E-05 | 1.8E-05 | 1.93E-26 | 2.52E-27 | 6.80E-09 |
| 3    | -1   | 1.8E-05 | 1.8E-05 | 1.61E-26 | 4.97E-27 | 7.04E-10 |
| 2    | 2    | 1.8E-05 | 1.8E-05 | 2.49E-08 | 5.04E-09 | 5.53E-09 |
| 2    | -2   | 1.8E-05 | 1.8E-05 | 1.27E-08 | 8.40E-09 | 8.03E-09 |
| 1    | 3    | 1.8E-05 | 1.8E-05 | 2.52E-26 | 5.66E-27 | 3.55E-09 |
| 1    | -3   | 1.8E-05 | 1.8E-05 | 1.63E-26 | 1.10E-26 | 8.64E-10 |
| 0    | 4    | 0.0E+00 | 3.6E-05 | 1.20E-08 | 1.45E-08 | 1.42E-08 |

No synchrotron motion. Left and right plots correspond to tune (22.40,20.75) and (21.39,21.43), respectively.

Beam charge density depends on $z$.

$$\lambda(z,s) = \frac{N_p}{\sqrt{2\pi\sigma_z}} \exp\left(-\frac{(z(s))^2}{2\sigma_z^2}\right) \quad (40)$$

The resonance amplitude $(J_R)$ and width $(\Delta J)$ are modulated by the synchrotron motion. The resonance structure moving and changing width in the phase space. Phase space lower amplitude than the resonance structure in Fig.9 is filled in chaotic area [4]. Therefore emittance growth is enhanced by the synchrotron motion. Figure 10 shows phase space plot with synchrotron motion.

Diffusion of $J$ characterizes emittance growth. To calculate of diffusion rate of $J$, initializing $\delta(J - J_0)$ and uniform $\phi$, an evolution of spread of $J$ is calculated. Figure 11 shows evolution of the spread of $J_x$. Top and bottom are given for without and with synchrotron motion. Diffusion of $J$ with
synchrotron motion is larger than those without synchrotron motion. All particles $J_x < 40\mu m$ diffuse for finite $\nu_s$, while limited particles $J_x = 30$ and 40 have large $\Delta J$, but not diffusive.

![Figure 11: Diffusion of $J_x$. Top is for no synchrotron motion. Bottom is for Synchrotron tune, $\nu_s = 0.002$, $z_0 = \sigma_z$.](image)

**SUMMARY**

- Emittance growth based on chaos near resonances.
- Tune shift and slope for space charge and lattice is evaluated.
- Resonance terms for space charge and lattice is evaluated.
- Resonance fixed point and width are determined by the tune slope and resonance strength.
- Simulation of a model based on the tune and resonance information is being performed.
- Emittance growth is evaluated by combination with Synchrotron motion.
- Simulations for Multi-resonances will be performed.
- Simulations taking into account of superperiodicity and errors of beta will be performed.

**REFERENCES**