Vlasov equation approach to space charge effects in isochronous cyclotrons

Antoine Cerfon
Courant Institute of Mathematical Sciences, NYU
with J. Guadagni and O. Bühler (CIMS NYU)
J.P. Freidberg and F.I. Parra (MIT PSFC)
**Motivation:** Spiraling in Isochronous Machines

**PSI Injector II**

PICN (Adam)  
OPAL-CYCL (Adelmann *et al.*)
**Motivation: Breakup in Isochronous Machines**

Small Isochronous Ring
(Pozdeyev et al.)

CYCIAE 100
(Bi et al.)
MOTIVATION

- Beam spiraling and beam breakup
  - Observed in PIC simulations
  - Indirect experimental observations
  - Single-particle explanation

- PIC simulations
  - Combine Newton’s equations with Maxwell’s equations
  - Strengths: Conceptually simple, reliable tool, moves the theoretical effort from physics to computer science
  - Weaknesses: Does not provide intuitive understanding, computationally costly

- Single particle picture: how to extend it for the nonlinear dynamics?

- Can we use continuum kinetic theory for an intuitive and analytic explanation of these effects?
OUTLINE

▶ Fluid theory of beam vortex motion
  ▶ Fluid model for highly intense proton beams
  ▶ Multiple time scale analysis and averaging procedure

▶ Beam stability
  ▶ Comparison with PIC simulations
  ▶ Isomorphism with Euler equations for a fluid
  ▶ Beam stability theorems
OUTLINE

- **Fluid theory of beam vortex motion**
  - Fluid model for highly intense proton beams
  - Multiple scale analysis and averaging procedure

- **Beam stability**
  - Comparison with PIC simulations
  - Isomorphism with Euler equations for a fluid
  - Beam stability theorems
Our fluid theory is based on two principles:

1. The principle of **maximal** geometric simplification:
   - Homogeneous magnetic field: $B = Be_z$
   - Non-relativistic coasting beam
   - Two-dimensional problem: $\partial/\partial z \equiv 0$ for all quantities

2. The principle of **minimal** beam physics simplification

We will derive all the results from kinetic theory and the Vlasov equation, using only one assumption:

The amplitude of the mismatch oscillations is small compared to the size of the proton beam (i.e. departure from laminar flow is small)
Laminar

Lab frame

Moving frame

Departure from laminar regime
How small is small?

- Let $a$ be the characteristic size of the beam; $n$ the beam density; $v_T$ the beam thermal velocity; $\omega_c$ the cyclotron frequency

- Relative size of space charge force and magnetic force

$$\left| \frac{E}{v \times B} \right| \sim \frac{e^2 an}{m e_0 v_T \omega_c} \sim \frac{e^2 n}{m e_0 \omega_c^2} \equiv \frac{\omega_p^2}{\omega_c^2}$$

where $\omega_p$ is the plasma frequency

- Define

$$\delta^2 = \frac{\omega_p^2}{\omega_c^2} = \frac{mn}{e_0 B^2}$$

- All cyclotrons and rings satisfy $\delta^2 \leq 1$, and most satisfy $\delta^2 \ll 1$

- We consider the regime in which the mismatch oscillation amplitude is of order $\delta a$
DERIVING FLUID EQUATIONS FOR THE BEAM

- Evolution of the beam distribution function \( f(x, v, t) \) in phase space given by the Vlasov equation:

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{e}{m} (E + v \times B) \cdot \nabla_v f = 0
\]

- Taking the integrals \( \int dv \) and \( \int vdv \) of this equation, and defining \( n \equiv \int f dv, nV = \int vfdv, \) and \( P = m \int (v - V)(v - V)fdv, \) we can obtain fluid-like equations:

\[
\frac{\partial n}{\partial t} + \nabla \cdot (nV) = 0 \quad \text{Continuity}
\]

\[
mn \left( \frac{\partial V}{\partial t} + V \cdot \nabla V \right) = en (E + V \times B) - \nabla \cdot P \quad \text{Momentum}
\]

- Well-known closure problem: too many unknowns and not enough equations
  ⇒ In general, need kinetic codes to solve the Vlasov equation
PRESSURE TENSOR

- When $\delta^2 \ll 1$ and the mismatch oscillation amplitude is of order $\delta a$, we can derive closed fluid equations.

- That’s because to lowest order in $\delta$ the tensor $\mathbf{P}$ has the form

\[
\mathbf{P} = p_\perp \mathbf{I} + (p_\parallel - p_\perp) \mathbf{e}_z \mathbf{e}_z = \begin{pmatrix}
p_\perp & 0 & 0 \\
0 & p_\perp & 0 \\
0 & 0 & p_\parallel
\end{pmatrix}
\]

- Proof: Compare all the terms to the magnetic term in the Vlasov equation, with $|v| \sim \delta v_T$:

\[
\begin{align*}
\frac{\mathbf{E}}{\mathbf{v} \times \mathbf{B}} & \sim \frac{aen}{\delta a \omega_c \epsilon_0 B} \sim \frac{\omega_p^2}{\delta \omega_c^2} \sim \delta \\
\frac{\mathbf{v} \cdot \nabla f}{e/m \mathbf{v} \times \mathbf{B} \cdot \nabla \mathbf{v} f} & \sim \frac{\omega_c}{\omega_c} \sim \delta \\
\frac{\partial f}{\partial t} & \sim \frac{v}{a \omega_c} \sim \delta
\end{align*}
\]

- To lowest order in $\delta$, the Vlasov equation therefore is

\[
\mathbf{v} \times \mathbf{B} \cdot \nabla \mathbf{v} f = 0 \quad \Rightarrow \frac{\partial f}{\partial \varphi} = 0 \quad (\varphi : \text{gyrophase})
\]
\[ \nabla \cdot P = \nabla p_\perp + (p_\parallel - p_\perp) e_z \cdot \nabla e_z + e_z \cdot \nabla (p_\parallel - p_\perp) e_z + (p_\parallel - p_\perp) \nabla \cdot e_z e_z \]
\[
= \nabla p_\perp + e_z \cdot \nabla (p_\parallel - p_\perp) e_z
\]

- In the radial-longitudinal plane, **only contribution is** \( \nabla p_\perp \)
- Fluid equations in the frame moving with the beam for the 2D motion in the radial-longitudinal plane:

\[
\frac{dn}{dt} + n \nabla \cdot v = 0 \quad \frac{dv}{dt} + v \times e_z = -\delta^2 \left( \nabla \phi + \frac{\alpha^2}{n} \nabla p_\perp \right)
\]
\[
\nabla^2 \phi = -n \quad \alpha^2 \equiv T_{\text{max}}/ma^2 \omega_p^2
\]

- **IMPORTANT:** The pressure term comes in as an exact gradient; as we prove later, this means that we **do not need an equation for the evolution of** \( p_\perp \)
MULTIPLE TIME SCALE ANALYSIS

▶ When $\delta^2 \ll 1$, the space charge time scale is much longer than the betatron time scale

▶ The particle motion is quasi-periodic

▶ This can be used to reduce the complexity of numerical simulations (e.g. PICS)

▶ We use it to derive simple fluid equations valid on the space charge time scale, using a multiple time scale analysis
Step 1: Each quantity $Q$ is assumed to vary according to the different time scales as follows:

$$Q(r, t) = Q(r, t_0, t_2, t_4, \ldots) = Q(r, t, \delta^2 t, \delta^4 t, \ldots)$$

t_0$ is the betatron time scale; $t_2 \sim \delta^2 t_0$ is the space charge time scale

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial t_0} + \delta^2 \frac{\partial Q}{\partial t_2} + \ldots$$

Step 2: Define averaging operation over the betatron time scale

$$\langle Q \rangle = \frac{1}{2\pi} \int_0^{2\pi} Q(r, t_0, t_2, \ldots) dt_0$$

Separate $Q$ into the sum of a rapidly oscillating part $\tilde{Q}$, and a slow monotonic evolution $\bar{Q}$:

$$Q(r, t_0, t_2, \ldots) = \tilde{Q}(r, t_0, t_2, \ldots) + \bar{Q}(r, t_2, \ldots)$$

where by definition $\langle \tilde{Q} \rangle = 0$
Step 3: Expand all quantities in $\delta$. The expansion corresponding to our assumptions is

\[
\begin{align*}
n &= \bar{n}_0 + \delta (\tilde{n}_1 + \bar{n}_1) + \delta^2 (\tilde{n}_2 + \bar{n}_2) + O(\delta^3) \\
p_\perp &= \bar{p}_0 + \delta (\tilde{p}_1 + \bar{p}_1) + \delta^2 (\tilde{p}_2 + \bar{p}_2) + O(\delta^3) \\
\phi &= \bar{\phi}_0 + \delta (\tilde{\phi}_1 + \bar{\phi}_1) + \delta^2 (\tilde{\phi}_2 + \bar{\phi}_2) + O(\delta^3) \\
v &= \delta \tilde{v}_1 + \delta^2 (\tilde{v}_2 + \bar{v}_2) + O(\delta^3)
\end{align*}
\]

Step 4: Plug this expansion into the fluid equations and solve order by order in $\delta$. For the density, we have

\[
\begin{align*}
\frac{\partial \tilde{n}_1}{\partial t_0} + \nabla \cdot (\bar{n}_0 \tilde{v}_1) &= 0 \quad O(\delta) \\
\frac{\partial \tilde{n}_2}{\partial t_0} + \frac{\partial \bar{n}_0}{\partial t_2} + \nabla \cdot [(\tilde{n}_1 + \bar{n}_1) \tilde{v}_1 + \bar{n}_0 (\tilde{v}_2 + \bar{v}_2)] &= 0 \quad O(\delta^2) \\
\frac{\partial \bar{n}_0}{\partial t_2} + \nabla \cdot (\langle \tilde{n}_1 \tilde{v}_1 \rangle + \bar{n}_0 \bar{v}_2) &= 0
\end{align*}
\]
\( \tilde{n}_1 \) and \( \tilde{v}_1 \) are given by the lowest order betatron motion: easy to compute \(^1\)

- Taking the momentum equation to \( O(\delta^2) \) and averaging it on the fast time scale, we find:

\[
\bar{v}_2 = \langle \tilde{v}_1 \cdot \nabla \tilde{v}_1 \rangle \times \mathbf{e}_z + \nabla \tilde{\phi}_0 \times \mathbf{e}_z + \frac{\alpha^2}{n_0} \nabla \tilde{p}_0 \times \mathbf{e}_z
\]

- Combining all the results, we find after some algebra \(^1\)

\[
\frac{\partial \tilde{n}_0}{\partial t_2} + \nabla \cdot (\tilde{n}_0 \nabla \tilde{\phi}_0 \times \mathbf{e}_z) + \alpha^2 \nabla \cdot [\nabla \times (\tilde{p}_0 \mathbf{e}_z)] = 0
\]

\[
\Leftrightarrow \frac{\partial \tilde{n}_0}{\partial t_2} + \nabla \tilde{\phi}_0 \times \mathbf{e}_z \cdot \nabla \tilde{n}_0 = 0
\]

- In our ordering, temperature effects do not play any effect on the slow time scale (at least to lowest order)

\(^1\)A.J. Cerfon, J.P. Freidberg, F.I. Parra, and T.A. Antaya, PRSTAB 16, 024202 (2013)
Beam dynamics due to space charge effects

- Final result:

\[
\frac{\partial \bar{n}_0}{\partial t} + \nabla \phi_0 \times e_z \cdot \nabla \bar{n}_0 = 0 \\
\nabla^2 \phi_0 = -\bar{n}_0
\]

- Describes the advection of the density profile in the velocity field \(E \times B/B^2\)

- Agrees with single-particle picture, and extends it to the nonlinear regime

- Our result is a first-principle derivation of Gordon’s\(^2\) intuition

- \(\delta^2\) only appears through \(t_2\). Bunches with different densities have identical behavior. Only difference: growth rates and frequency scale linearly with \(n\)

ExB advection
OUTLINE

▶ Fluid theory of beam vortex motion
  ▶ Fluid model for highly intense proton beams
  ▶ Multiple scale analysis and averaging procedure
  ▶ Isomorphism with Euler equations for a fluid

▶ Beam stability
  ▶ Comparison with PIC simulations
  ▶ Isomorphism with Euler equations for a fluid
  ▶ Beam stability theorems
Is our model relevant?

Our simulation

PICS simulation

- Good agreement, even at $\delta^2 = 0.8$
- Geometrical effects play a very limited role
**ISOMORPHISM WITH 2D EULER EQUATIONS**

**Beam vortex dynamics**

\[
\frac{\partial n}{\partial t} + \nabla \phi \times e_z \cdot \nabla n = 0
\]

\[
\nabla^2 \phi = -n
\]

**2D incompressible Euler**

\[
\frac{\partial \omega}{\partial t} + \nabla \psi \times e_z \cdot \nabla \omega = 0
\]

\[
\nabla^2 \psi = -\omega
\]

\(n\): bunch density; \(\phi\): electrostatic potential

\(\omega\): z-directed vorticity; \(\psi\): stream function for the flow

- Isomorphism recognized a long time ago in a slightly different context\(^3\)

- We proved that the isomorphism holds even for finite temperature beams

- We can use decades old fluid dynamics results to determine/understand the stability of bunch distributions

---

Stability of Round Beams

- Radial density distributions automatically satisfy the equations.
- Well-known results from fluid theory of radially symmetric vortex patches:
  - If \( n(r) \) is **monotonically decreasing**, the bunch is **nonlinearly stable** to nonsymmetric density perturbations.
  - Hollow density profiles can be unstable to these perturbations.

Gaussian \( n(r), \delta^2 = 0.8 \)
ELLIPTI C BUNCHES WITH UNIFORM DENSITY

- Classical case in fluid dynamics: uniform density profile
  Call $a$ the semi-major axis and $b$ the semi-minor axis
- If $a/b < 3$, bunch is linearly and non-linearly stable to edge perturbations
  If $a/b > 3$, bunch is linearly and non-linearly unstable to edge perturbations
- Instability is a potential mechanism for beam breakup

Uniform $n$, $\delta^2 = 0.2$, $a/b = 20$
**ELLiptic Bunches with Smooth Density Profile**

- More complicated case. Answer depends on the smoothness of the profile
- For reasonably smooth profile, "axisymmetrization principle"\(^4\) even for \(a/b < 3\)

**BEAM SPIRALING A.K.A. AXISYMMETRIZATION**

Our simulation

OPAL simulation

- In PSI Injector II, formation of a round stable core after 40 turns (good)
- In machines with lower $\delta^2$, core and halo take longer to form
  Potentially bad situation if low density halo forms with high energy
SUMMARY

- Space-charge forces are small relative to magnetic forces by the ratio $\delta^2 = \omega_p^2/\omega_c^2 \leq 1$

- The scale separation between betatron and space charge time scales can be advantageously used to reduce the complexity of kinetic calculations and reduce computational time

- When the mismatch amplitude is small by $\delta$ compared to the typical size of the beam, a fluid model of the beam can be rigourously derived from the Vlasov equation

- In the fluid model, beam spiraling is a consequence of the advection of the beam in the $E \times B$ velocity field

- Bunches behave like isolated vortex distributions in the 2D incompressible Euler equations

- This physical picture is in quantitative agreement with PIC simulations
ONGOING AND FUTURE WORK

▶ Include accelerating gaps. Design accelerating voltage shape/phase to counter spiraling?

▶ Allow arbitrary departure from laminar regime
  ▶ Requires numerically solving a kinetic equation
  ▶ Idea: use scale separation between betatron and space charge time scales to simplify Vlasov equation
  ▶ Write a reduced continuum kinetic code

▶ Relativistic regime (Adelmann)

▶ Consider realistic magnetic field configurations and 3D effects

▶ Develop collaboration with experimentalists and PIC developers
Suggestions you might have? Lets discuss them!